Perturbation models for interfacial cracks

A. Piccolroaz[1] G. Mishuris[2] A.B. Movchan[3]

[1] Department of Mechanical and Structural Engineering
University of Trento

[2] Institute of Mathematics and Physics
Aberystwyth University
and
Wales Institute of Mathematical and Computational Sciences

[3] Department of Mathematical Sciences
University of Liverpool

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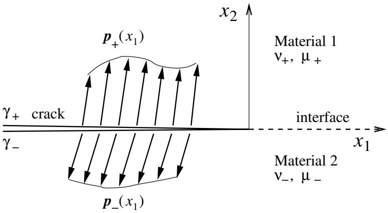
Outline of the talk

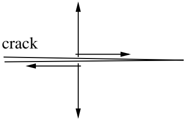
1. Problem formulation
   - 2D case
   - 3D case
2. Fundamental solutions
   - Fundamental solutions in linear elasticity and fracture mechanics
   - Definition of weight functions
3. New fundamental solutions for interfacial cracks
   - 2D case
   - 3D case
   - Stress Intensity Factors
4. Illustrative examples
   - Computation of SIFs
   - 2D perturbation problem

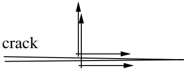


- Bi-material plane
- Perfect interface: $[\boldsymbol{u}] = 0$, $[\boldsymbol{t}_2] = 0$, for $x_1 > 0$.
- 2D half-plane crack along the interface
- The loading is self-balanced but NOT symmetrical:

$$\int_{\gamma_+} \boldsymbol{p}_+ + \int_{\gamma_-} \boldsymbol{p}_- = 0, \qquad \int_{\gamma_+} \boldsymbol{x} \wedge \boldsymbol{p}_+ + \int_{\gamma_-} \boldsymbol{x} \wedge \boldsymbol{p}_- = 0,$$

Symmetric and Skew-symmetric components





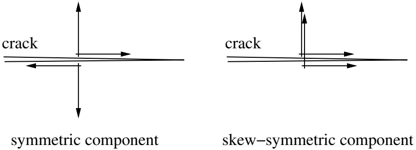
Symmetric load:

$$\langle \boldsymbol{p} \rangle (x_1) = \frac{\boldsymbol{p}_+(x_1) + \boldsymbol{p}_-(x_1)}{2}$$

Skew-symmetric load:

$$[\![\boldsymbol{p}]\!] (x_1) = \boldsymbol{p}_+(x_1) - \boldsymbol{p}_-(x_1)$$

Symmetric and Skew-symmetric components



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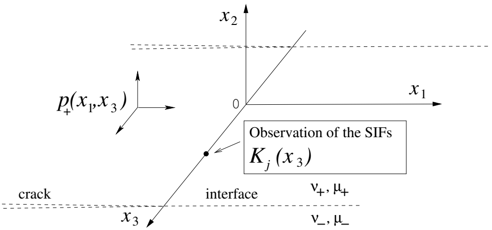
$$[\![\boldsymbol{p}]\!](x_1) = \boldsymbol{p}_+(x_1) - \boldsymbol{p}_-(x_1)$$

Questions

- Does the skew-symmetric component produce any stress concentration at the crack tip?
- Can we compute the SIFs (and possibly higher-order terms) for a general asymmetrical distribution of forces?

Geometry of the model: the 3D case



- Bi-material space
- The loading is a function of x_3:

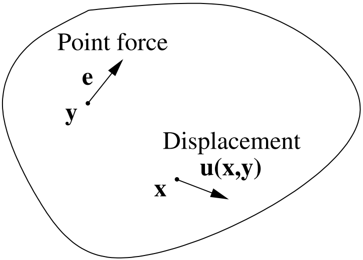
$$\langle \boldsymbol{p} \rangle (x_1, x_3) = \frac{\boldsymbol{p}_+(x_1, x_3) + \boldsymbol{p}_-(x_1, x_3)}{2}, \quad [\![\boldsymbol{p}]\!](x_1, x_3) = \boldsymbol{p}_+(x_1, x_3) - \boldsymbol{p}_-(x_1, x_3)$$

- The SIFs are functions of x_3:

$$\begin{cases} K(x_3) = K_{\mathrm{I}}(x_3) + iK_{\mathrm{II}}(x_3) \quad \text{[complex SIF]} \\ K_{\mathrm{III}} = K_{\mathrm{III}}(x_3) \end{cases}$$

Green's function

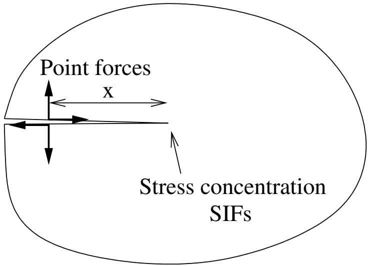
Solution of the displacement field at point $\boldsymbol{x}$ produced by a unit concentrated body force $\boldsymbol{e}$ located at point $\boldsymbol{y}$.



$$u_i(\boldsymbol{x},\boldsymbol{y}) = G_{ij}(\boldsymbol{x},\boldsymbol{y})e_j$$

$$G_{ij}(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{16\pi\mu(1-\nu)r}[(3-4\nu)\delta_{ij} + r_{,i}\, r_{,j}\,], \quad \boldsymbol{r} = \boldsymbol{x} - \boldsymbol{y}, \quad r = |\boldsymbol{r}|$$

Fundamental solutions in linear fracture mechanics



$$K = \sqrt{\frac{2}{\pi}}(1+i)x^{-1/2}$$

$$K_{\mathrm{I}} = \sqrt{\frac{2}{\pi}}x^{-1/2}, \quad K_{\mathrm{II}} = \sqrt{\frac{2}{\pi}}x^{-1/2}$$

Fundamental solutions in linear fracture mechanics are known as weight functions.

Definition of Weight Functions (Bueckner, 1970)

Definition

SIF associated with concentrated point forces applied on the crack faces.

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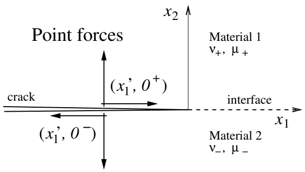
2D case



2D weight function: $w = w(x_1')$

$$K = \int_{-\infty}^{0} w(x_1') \underbrace{p(x_1')}_{\text{distributed load}} dx_1'$$

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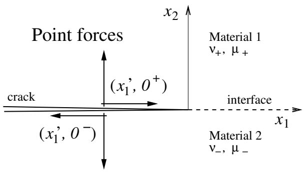

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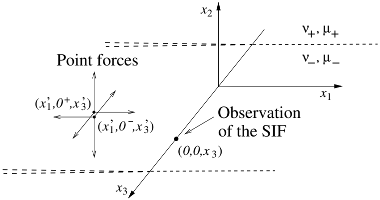

3D weight function: $w = w(x_1', x_3', x_3)$

$$K(x_3) = \int_{-\infty}^{\infty} \int_{-\infty}^{0} w(x_1', x_3', x_3) \underbrace{p(x_1', x_3')}_{\text{distributed load}} dx_1' dx_3'$$

Generalization of Weight Functions (Willis-Movchan, 1995)

Definition

Singular solution to the elastic crack problem with homogeneous boundary conditions (traction-free crack faces): displacement field $\boldsymbol{U}(x_1, x_2)$, stress field $\boldsymbol{\Sigma}(x_1, x_2)$.

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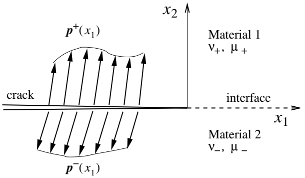

Model domain for the **physical solution**
The crack lies on the left-hand side

$$\boldsymbol{u} \sim \sum_{j=1}^{3} K_j r^{1/2+i\epsilon} f_j(\theta)$$

$$\boldsymbol{\sigma} \sim \sum_{j=1}^{3} K_j r^{-1/2+i\epsilon} g_j(\theta)$$

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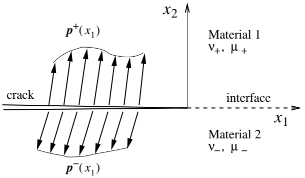

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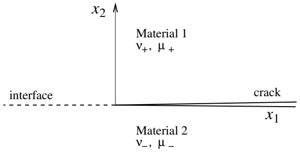

Model domain for the singular solutions
The crack lies on the right-hand side

$$\boldsymbol{U} \sim \sum_{j=1}^{3} K_j r^{-1/2+i\epsilon} F_j(\theta)$$

$$\boldsymbol{\Sigma} \sim \sum_{j=1}^{3} K_j r^{-3/2+i\epsilon} G_j(\theta)$$

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- derivation of SIFs associated to concentrated forces on the crack faces (Bueckner weight functions):

$$w(x_1', x_3', x_3) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} [\![\overline{U}]\!](\beta, \lambda) e^{i\beta x_1'} e^{i\lambda(x_3' - x_3)} d\beta d\lambda,$$

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- derivation of SIFs associated to a general asymmetrical loading on the crack faces:

$$K(x_3) = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \left\{ [\![\overline{U}]\!](\beta, \lambda) \langle \overline{p} \rangle (\beta, \lambda) + \langle \overline{U} \rangle (\beta, \lambda) [\![\overline{p}]\!](\beta, \lambda) \right\} e^{-i\lambda x_3} d\beta d\lambda,$$

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- evaluation of the constants near high-oder terms in the asymptotics of the solution

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$$w(x_1', x_3', x_3) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} [\![\overline{U}]\!](\beta,\lambda) e^{i\beta x_1'} e^{i\lambda(x_3'-x_3)} d\beta d\lambda,$$

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- evaluation of the constants near high-oder terms in the asymptotics of the solution
- solution of perturbation problems (load and/or geometrical perturbations)

- Plane strain (Mode I and II)
  - **Symmetric weight function matrix**:

$$\begin{cases} [\![\boldsymbol{U}]\!](x_1) = \dfrac{1}{2d_0\sqrt{2\pi x_1}} \left\{ \dfrac{x_1^{-i\epsilon}}{c_1^+}\boldsymbol{\mathcal{B}} + \dfrac{x_1^{i\epsilon}}{c_1^-}\boldsymbol{\mathcal{B}}^\top \right\} & \text{for } x_1 > 0, \\ [\![\boldsymbol{U}]\!](x_1) = 0 & \text{for } x_1 < 0, \end{cases} \qquad \boldsymbol{\mathcal{B}} = \begin{bmatrix} 1 & -i \\ i & 1 \end{bmatrix}$$

  - **Skew-symmetric weight function matrix**:

$$\begin{cases} \langle \boldsymbol{U} \rangle(x_1) = \dfrac{\alpha}{2}[\![\boldsymbol{U}]\!](x_1) & \text{for } x_1 > 0, \\ \langle \boldsymbol{U} \rangle(x_1) = -i\dfrac{\alpha(d_* - \gamma_*)}{4d_0^3\sqrt{-2\pi x_1}} \left\{ \dfrac{(-x_1)^{-i\epsilon}}{c_1^+}\boldsymbol{\mathcal{B}} - \dfrac{(-x_1)^{i\epsilon}}{c_1^-}\boldsymbol{\mathcal{B}}^\top \right\} & \text{for } x_1 < 0, \end{cases}$$

- Antiplane shear (Mode III):
  - **Symmetric and skew-symmetric weight functions:**

$$[\![U_3]\!](x_1) = \begin{cases} \dfrac{1-i}{\sqrt{2\pi}} x_1^{-1/2}, & \text{for } x_1 > 0, \\ 0, & \text{for } x_1 < 0, \end{cases} \quad , \quad \langle U_3 \rangle = \frac{\eta}{2}[\![U_3]\!],$$

The 3D case: new fundamental solutions for interfacial cracks

- **Symmetric weight function**:

$$[\![\overline{\boldsymbol{U}}]\!]^+(\beta,\gamma) = \frac{\lambda}{\sqrt{|\lambda|}\sqrt{\beta+i|\lambda|}} \begin{bmatrix} F_{11}(\beta,\lambda) & F_{12}(\beta,\lambda) & F_{13}(\beta,\lambda) \\ F_{21}(\beta,\lambda) & F_{22}(\beta,\lambda) & F_{23}(\beta,\lambda) \\ F_{31}(\beta,\lambda) & F_{32}(\beta,\lambda) & F_{33}(\beta,\lambda) \end{bmatrix}$$

- **Skew-symmetric weight function**:

$$2\langle\overline{\boldsymbol{U}}\rangle(\beta,\lambda) = \frac{1}{\beta+i|\lambda|}\boldsymbol{C}_1^-(\beta)\overline{\boldsymbol{\Sigma}}^- + \left(\alpha\boldsymbol{\Pi}_{12} + \eta\boldsymbol{\Pi}_3 + \frac{1}{\beta-i|\lambda|}\boldsymbol{C}_2^+(\beta)\right)[\![\overline{\boldsymbol{U}}]\!]^+,$$

$$\boldsymbol{C}_1^-(\beta) = \frac{ib\alpha(d_* - \gamma_*)}{\beta-i|\lambda|}\begin{bmatrix} 0 & -\beta & 0 \\ \beta & 0 & \lambda \\ 0 & -\lambda & 0 \end{bmatrix}, \quad \boldsymbol{\Pi}_{12} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$\boldsymbol{C}_2^+(\beta) = \frac{f-\alpha e}{(b+e)(\beta+i|\lambda|)}\begin{bmatrix} \lambda^2 & 0 & -\beta\lambda \\ 0 & 0 & 0 \\ -\beta\lambda & 0 & -\lambda^2 \end{bmatrix}, \quad \boldsymbol{\Pi}_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

It is possible to derive the 2D weight functions as limiting case $\lambda \to 0$.

The 2D case: integral formula for the computation of SIFs

- Plane strain (Mode I and II): the complex SIF, $\boldsymbol{K} = [K, K^*]^\top$, is given by:

$$\boldsymbol{K} = -i\mathcal{M}_1^{-1} \lim_{x_1' \to 0} \int_{-\infty}^{0} \Big\{ \underbrace{[\![\boldsymbol{U}]\!]^\top (x_1' - x_1) \boldsymbol{R} \langle \boldsymbol{p} \rangle (x_1)}_{\text{symmetric part}} + \underbrace{\langle \boldsymbol{U} \rangle^\top (x_1' - x_1) \boldsymbol{R} [\![\boldsymbol{p}]\!] (x_1)}_{\text{skew-symmetric part}} \Big\} dx_1,$$

$[\![\boldsymbol{U}]\!]$ symmetric weight function, $\langle \boldsymbol{U} \rangle$ skew-symmetric weight function

$\langle \boldsymbol{p} \rangle$ symmetric load, $[\![\boldsymbol{p}]\!]$ skew-symmetric load

- Antiplane shear (Mode III):

$$K_{III} = -(1+i) \lim_{x_1' \to 0^+} \int_{-\infty}^{0} \{ \underbrace{[\![U_3]\!](x_1' - x_1) \langle p_3 \rangle (x_1)}_{\text{symmetric part}} + \underbrace{\langle U_3 \rangle (x_1' - x_1) [\![p_3]\!](x_1)}_{\text{skew-symmetric part}} \} dx_1,$$

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Answer

"The skew-symmetric component does produce stress concentration at the crack tip!"

Material parameters involved in the interfacial crack problem

Symmetric weight function $[\![\boldsymbol{U}]\!]$	Skew-symmetric weight function $\langle \boldsymbol{U} \rangle$
$b = \dfrac{1-\nu_+}{\mu_+} + \dfrac{1-\nu_-}{\mu_-}$	$b\alpha = \dfrac{1-\nu_+}{\mu_+} - \dfrac{1-\nu_-}{\mu_-}$
$d = \dfrac{1-2\nu_+}{2\mu_+} - \dfrac{1-2\nu_-}{2\mu_-}$	$b\gamma = \dfrac{1-2\nu_+}{2\mu_+} + \dfrac{1-2\nu_-}{2\mu_-}$
$e = \dfrac{\nu_+}{\mu_+} + \dfrac{\nu_-}{\mu_-}$	$f = \dfrac{\nu_+}{\mu_+} - \dfrac{\nu_-}{\mu_-}$
$d_* = \dfrac{d}{b} = \dfrac{\mu_-(1-2\nu_+) - \mu_+(1-2\nu_-)}{2\mu_-(1-\nu_+) + 2\mu_+(1-\nu_-)}$	$\gamma_* = \dfrac{\gamma}{\alpha} = \dfrac{\mu_-(1-2\nu_+) + \mu_+(1-2\nu_-)}{2\mu_-(1-\nu_+) - 2\mu_+(1-\nu_-)}$
$\alpha = \dfrac{\mu_-(1-\nu_+) - \mu_+(1-\nu_-)}{\mu_-(1-\nu_+) + \mu_+(1-\nu_-)}$	$\eta = \dfrac{\mu_- - \mu_+}{\mu_- + \mu_+}$

Dundurs parameters

Computation of SIFs for a plane strain deformation

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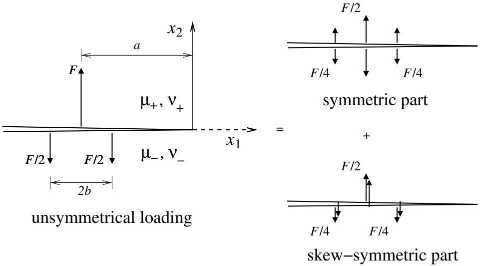
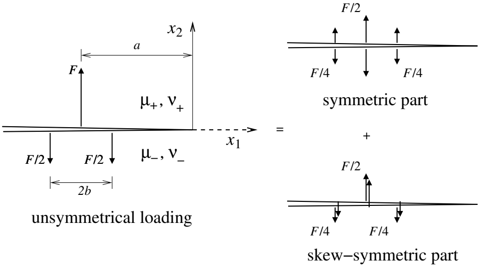


- Point forces in a “three-point” configuration
- The load is self-balanced in terms of both principal force and principal moment vectors
- The load is not symmetrical
- Symmetric and skew-symmetric loads (δ is the Dirac delta function):

Computation of SIFs for a plane strain deformation



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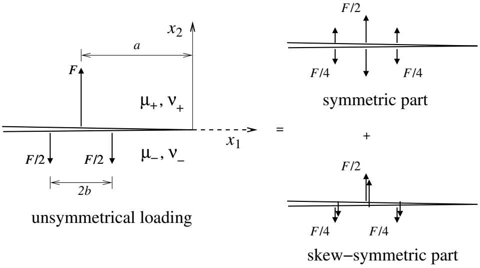
$$\langle p \rangle(x_1) = -F/2\delta(x_1 + a) - F/4\delta(x_1 + a + b) - F/4\delta(x_1 + a - b),$$

$$[\![p]\!](x_1) = -F\delta(x_1 + a) + F/2\delta(x_1 + a + b) + F/2\delta(x_1 + a - b).$$

Computation of SIFs for a plane strain deformation



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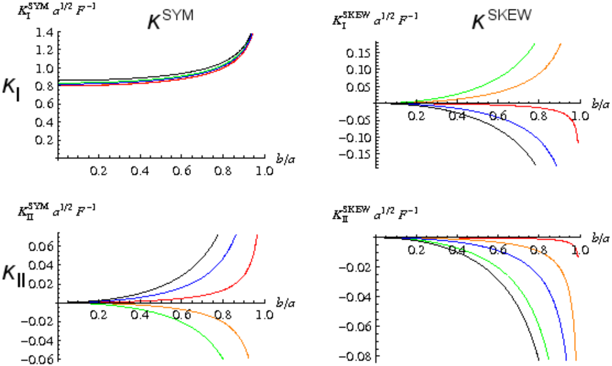
$$[\![p]\!](x_1) = -F\delta(x_1 + a) + F/2\delta(x_1 + a + b) + F/2\delta(x_1 + a - b).$$

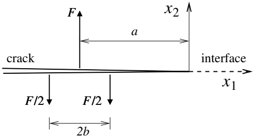
Complex SIF: $K = K^{\mathrm{SYM}} + K^{\mathrm{SKEW}}$

$$K^{\mathrm{SYM}} = F\sqrt{\frac{2}{\pi}}\cosh(\pi\epsilon)a^{-1/2-i\epsilon}\left\{\frac{1}{2} + \frac{1}{4}(1 + b/a)^{-1/2-i\epsilon} + \frac{1}{4}(1 - b/a)^{-1/2-i\epsilon}\right\}$$

$$K^{\mathrm{SKEW}} = \alpha F\sqrt{\frac{2}{\pi}}\cosh(\pi\epsilon)a^{-1/2-i\epsilon}\left\{\frac{1}{2} - \frac{1}{4}(1 + b/a)^{-1/2-i\epsilon} - \frac{1}{4}(1 - b/a)^{-1/2-i\epsilon}\right\}$$

K vs. b/a





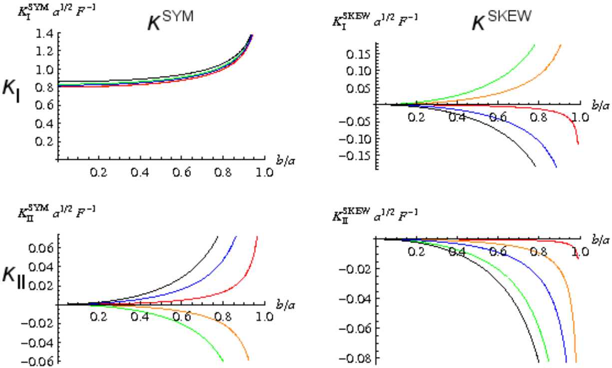
$\nu_+ = 0.2, \quad \nu_- = 0.3$

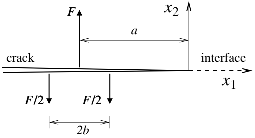
$\eta = (\mu_- - \mu_-)/(\mu_- + \mu_-)$

$\eta =$		
	−0.99	(green)
	−0.5	(orange)
	0.	(red)
	0.5	(blue)
	0.99	(black)

Numerical results

K vs. b/a





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	0.99	(black)

Comments:

- K_{II}^{SYM} and K_{II}^{SKEW} are NOT identically zero
- $K_I^{SKEW} = 0$ and $K_{II}^{SKEW} = 0$ for $b/a = 0$, but both increase when increasing b/a
- SIFs blow up as $b/a \to 1$

Symmetric and **skew-symmetric** weight functions can be efficiently used for:

High-order terms and perturbation problems

Symmetric and **skew-symmetric** weight functions can be efficiently used for:

- Evaluation of the constants near high-order terms:

$$\sigma_{22}(x_1,0) + i\sigma_{12}(x_1,0) \sim \frac{K}{\sqrt{2\pi}} x_1^{-1/2+i\epsilon} + \frac{A}{\sqrt{2\pi}} x_1^{1/2+i\epsilon}, \quad x_1 \to 0^+$$

$$\boldsymbol{A} = \boldsymbol{\mathcal{M}}_2^{-1} \lim_{x_1' \to 0} \int_{-\infty}^{0} \left\{ \frac{d[\![\boldsymbol{U}]\!]^\top (x_1' - x_1)}{dx_1} \boldsymbol{R} \langle \boldsymbol{p} \rangle (x_1) + \frac{d\langle \boldsymbol{U} \rangle^\top (x_1' - x_1)}{dx_1} \boldsymbol{R} [\![\boldsymbol{p}]\!](x_1) \right\} dx_1 .$$

High-order terms and perturbation problems

Symmetric and **skew-symmetric** weight functions can be efficiently used for:

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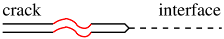
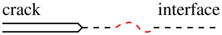
$$\boldsymbol{A}=\mathcal{M}_2^{-1}\lim_{x_1'\to 0}\int_{-\infty}^{0}\left\{\frac{d[\![\boldsymbol{U}]\!]^\top(x_1'-x_1)}{dx_1}\boldsymbol{R}\langle\boldsymbol{p}\rangle(x_1)+\frac{d\langle\boldsymbol{U}\rangle^\top(x_1'-x_1)}{dx_1}\boldsymbol{R}[\![\boldsymbol{p}]\!](x_1)\right\}dx_1.$$

- Solution of perturbation problems (load and/or geometry):

Uniform advance of the crack along the interface

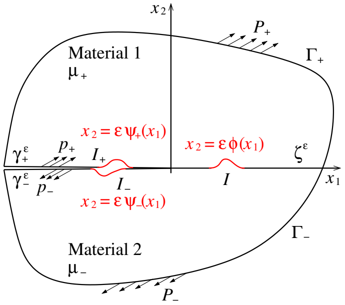


Out-of-plane perturbation of the crack faces



Out-of-plane perturbation of the interface

Perturbation of Mode III interfacial cracks



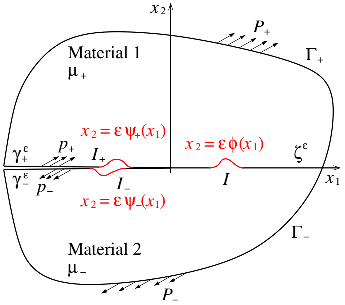
- Bi-material body subject to antiplane shear
- Asymmetrical loading on the crack faces
- Perturbations of crack faces and interface:

$$x_2 = \varepsilon\psi_\pm(x_1), \quad x_2 = \varepsilon\phi(x_1)$$

- SIF for the unperturbed problem ($\varepsilon = 0$):

$$K_{\rm III}^{(0)} = -\sqrt{\frac{2}{\pi}} \int_{-\infty}^{0} \left\{ \langle p \rangle(x_1) + \frac{\eta}{2} [\![p]\!](x_1) \right\} (-x_1)^{-1/2} dx_1$$

Perturbation of Mode III interfacial cracks



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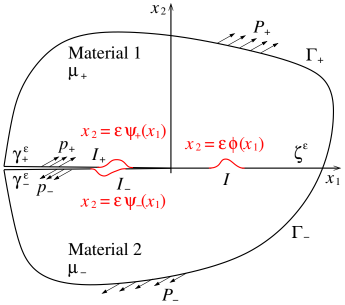
$$\Delta u^\pm(x_1,x_2) = 0, \quad \text{b.c.}\begin{cases}\mu_\pm\dfrac{\partial u^\pm}{\partial n^\pm} = P_\pm & \text{on } \Gamma_\pm\\[2ex] \mu_\pm\dfrac{\partial u^\pm}{\partial x_2} = p_\pm & \text{on } \gamma_\pm^\epsilon\end{cases} \qquad \text{i.c.}\begin{cases}u^+ = u^- & \text{on } \zeta^\epsilon\\[2ex] \mu_+\dfrac{\partial u^+}{\partial n} = \mu_-\dfrac{\partial u^-}{\partial n} & \text{on } \zeta^\epsilon\end{cases}$$



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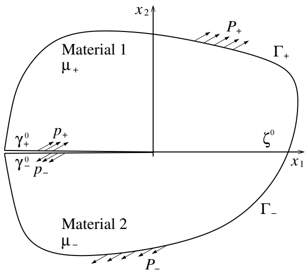
We assume that the functions $\psi_{\pm}, \phi$ and their derivatives vanish within a finite neighbourhood of the crack tip (regular perturbation of the boundary):

$$u^{\pm}(x_1, x_2, \varepsilon) = u^{\pm}_{(0)}(x_1, x_2) + \varepsilon u^{\pm}_{(1)}(x_1, x_2) + O(\varepsilon^2), \quad \varepsilon \to 0$$

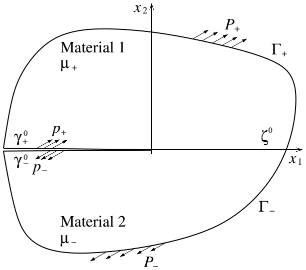
$$K_{\rm III} = K_{\rm III}^{(0)} + \varepsilon K_{\rm III}^{(1)} + O(\varepsilon^2), \quad \varepsilon \to 0$$



Model problem for $u^{\pm}_{(0)}(x_1, x_2)$

Solved by use of the Mellin transform:

$$\tilde{u}^{\pm}_{(0)}(s,\theta) = \frac{(\tilde{p}_+ - \tilde{p}_-)\cos(s\theta)}{(\mu_+ + \mu_-)s\sin(\pi s)} - \frac{(\mu_-\tilde{p}_+ + \mu_+\tilde{p}_-)\sin(s\theta)}{\mu_\pm(\mu_+ + \mu_-)s\cos(\pi s)}$$



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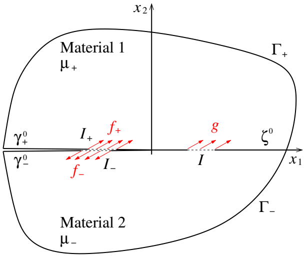


Model problem for $u^{\pm}_{(1)}(x_1, x_2)$

Effective loading on the crack faces:

$$f_\pm(x_1) := \mu_\pm \frac{\partial}{\partial x_1}\left(\psi_\pm(x_1)\frac{\partial u^{\pm}_{(0)}}{\partial x_1}\right)$$

Prescribed discontinuity of displacements along the interface:

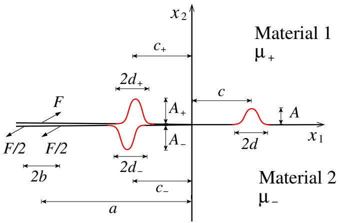
$$g(x_1) := -\phi(x_1)\left(\frac{\partial u^{+}_{(0)}}{\partial x_2} - \frac{\partial u^{-}_{(0)}}{\partial x_2}\right)$$

Integral formula for $K_{\rm III}^{(1)}$

Knowing the **symmetric** and **skew-symmetric** weight functions we can derive explicitly the first order variation of SIFs:

$$K_{\rm III}^{(1)} = \underbrace{-\sqrt{\frac{2}{\pi}}\int_{-\infty}^{0}(-x_1)^{-1/2}\frac{\partial}{\partial x_1}\left\{\langle f\rangle(x_1)+\frac{\eta}{2}[\![f]\!](x_1)\right\}dx_1}_{\text{perturbation of crack faces}} + \underbrace{\frac{\mu_+\mu_-}{\sqrt{2\pi}(\mu_+ + \mu_-)}\int_0^{\infty} g(x_1)x_1^{-3/2}dx_1}_{\text{perturbation of interface}}$$

Perturbation of crack faces and interface:



Loading:

$$\langle p\rangle(x_1) = \frac{1}{2}\delta(x_1+a) + \frac{1}{4}\delta(x_1+a+b) + \frac{1}{4}\delta(x_1+a-b)$$

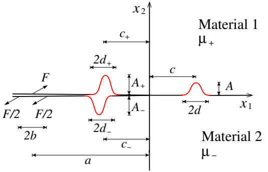
$$[\![p]\!](x_1) = \delta(x_1+a) - \frac{1}{2}\delta(x_1+a+b) - \frac{1}{2}\delta(x_1+a-b)$$

Perturbations:

$$\psi_\pm(x_1) = \pm\frac{A_\pm}{d_\pm^4}(x_1+c_\pm+d_\pm)^2(x_1+c_\pm-d_\pm)^2$$

$$\phi(x_1) = \frac{A}{d^4}(x_1+c+d)^2(x_1+c-d)^2$$

Numerical results

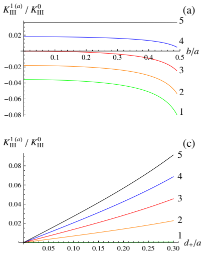


$$\eta = (\mu_- - \mu_-)/(\mu_- + \mu_-)$$

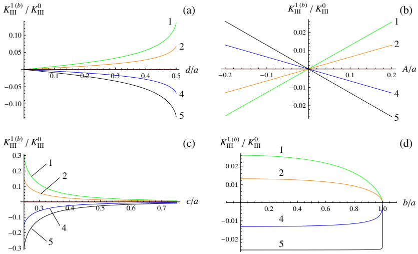
$\eta =$		
	-0.99	(green)
	-0.5	(orange)
	$0.$	(red)
	0.5	(blue)
	0.99	(black)

Numerical results for different material and geometrical parameters:

Perturbation of the crack faces:



Perturbation of the interface:



Summary

- Fundamental solutions in linear fracture mechanics
- New fundamental solutions for interfacial crack problems:
  - Symmetric and skew-symmetric weight functions for both 2D and 3D cases
  - Skew-symmetric loading does produce stress concentration at the crack tip
- How to use weight functions for:
  - Computation of SIFs
  - Computation of higher-order terms
  - Solve perturbation problems

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