

Mode III crack advancing in a bimaterial plane with finite array of defects

Weight function and dipole matrix approach

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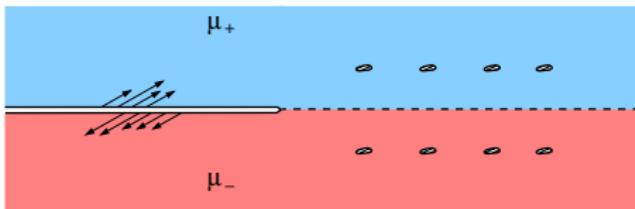
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The problem and motivation



Crack advancing in a **bimaterial** plane with finite array of defects

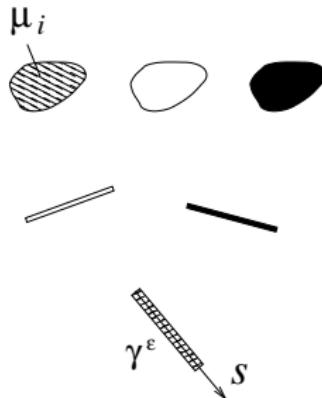
- What is the effect of the defects on the propagation of the crack?
Amplification/shielding effects?
- Can we arrange the defects in such a way to **stop the propagation** of the crack?
- Vice versa, can we arrange the defects in such a way to make the crack **propagate up to the end of the array**?

New challenges:

- Interfacial crack
- **Weight function** for an interfacial crack
- **Singular perturbation** procedure related to small defects
- **Singular perturbation** procedure related to small advance of the crack

Type of defects

- Small elastic inclusions
- Small voids
- Small rigid inclusions
- Microcracks
- Rigid line inclusions



Line defect with imperfect bonding:

- Line defect with **soft bonding** (stiffness κ): $[\![\sigma]\!](s) = 0, \quad \sigma(s) = \kappa [\![u]\!](s)$

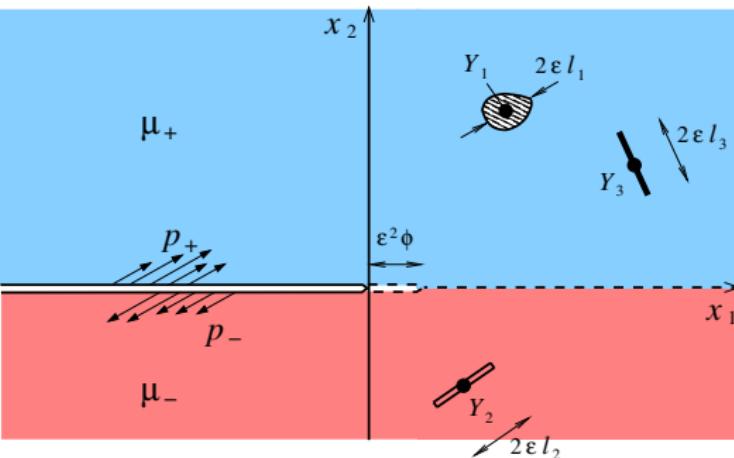
$$\begin{cases} \kappa = 0 & \Rightarrow \quad \sigma(s) = 0 \quad \text{microcrack} \\ \kappa = \infty & \Rightarrow \quad [\![u]\!](s) = 0 \quad \text{perfect bonding (no defect)} \end{cases}$$

- **Stiff** line defect (stiffness κ): $[\![u]\!](s) = 0, \quad [\![\sigma]\!](s) + \kappa \left. \frac{\partial^2 u}{\partial s^2} \right|_{\gamma^\epsilon} = 0$

$$\begin{cases} \kappa = 0 & \Rightarrow \quad [\![\sigma]\!](s) = 0 \quad \text{no defect} \\ \kappa = \infty & \Rightarrow \quad \left. \frac{\partial^2 u}{\partial s^2} \right|_{\gamma^\epsilon} = 0 \quad \text{rigid line inclusion} \end{cases}$$

Problem formulation

Bimaterial plane with a dominant crack along the interface and small defects:



Singular perturbations:

- ① Elastic inclusion
- ② Microcrack
- ③ Rigid line inclusion

Singular perturbation:

- ① Crack advance

Small parameter ε :

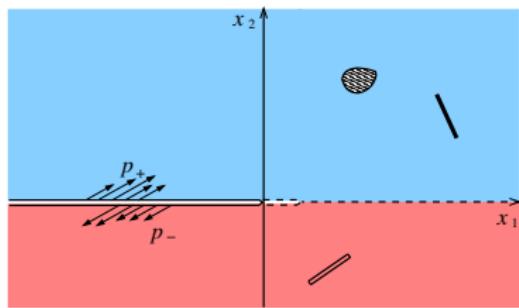
diameter of defect $2\varepsilon l$, $\varepsilon \ll 1$

Assumptions:

- ① Mode III deformation
- ② Loading on the crack surfaces
- ③ Perfect interfaces (continuity of displacements and tractions)
- ④ The composite is dilute (neglect interactions between small defects)
- ⑤ Stable quasi-static propagation (neglect inertia terms)

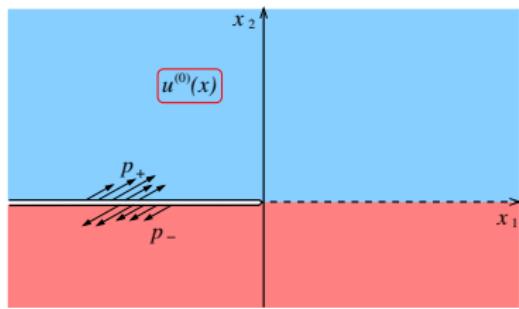
The asymptotic ansatz

$$u(\mathbf{x}, \varepsilon) = u^{(0)}(\mathbf{x}) + \varepsilon \sum_{j=1}^3 W_j(\xi_j) + \varepsilon^2 \sum_{j=1}^3 u^{(j)}(\mathbf{x}) + \varepsilon^2 v(\mathbf{x}, \phi) + o(\varepsilon^2), \quad \varepsilon \rightarrow 0$$



The asymptotic ansatz

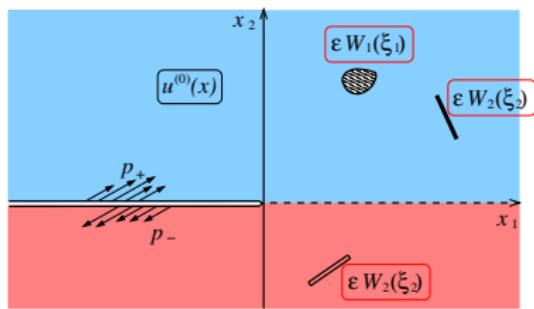
$$u(\mathbf{x}, \varepsilon) = \boxed{u^{(0)}(\mathbf{x})} + \varepsilon \sum_{j=1}^3 W_j(\boldsymbol{\xi}_j) + \varepsilon^2 \sum_{j=1}^3 u^{(j)}(\mathbf{x}) + \varepsilon^2 v(\mathbf{x}, \phi) + o(\varepsilon^2), \quad \varepsilon \rightarrow 0$$



➊ Solution of the unperturbed problem ($\varepsilon = 0$)

The asymptotic ansatz

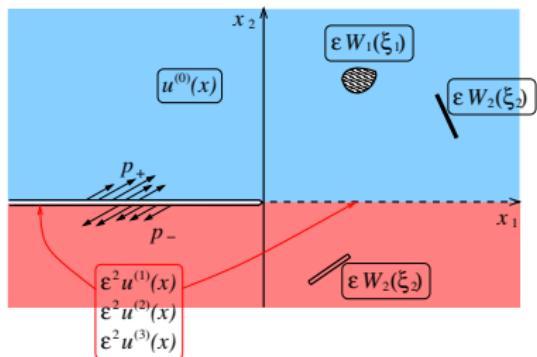
$$u(\mathbf{x}, \varepsilon) = u^{(0)}(\mathbf{x}) + \boxed{\varepsilon \sum_{j=1}^3 W_j(\xi_j)} + \varepsilon^2 \sum_{j=1}^3 u^{(j)}(\mathbf{x}) + \varepsilon^2 v(\mathbf{x}, \phi) + o(\varepsilon^2), \quad \varepsilon \rightarrow 0$$



- ➊ Solution of the unperturbed problem ($\varepsilon = 0$)
- ➋ Boundary layers concentrated near the defects

The asymptotic ansatz

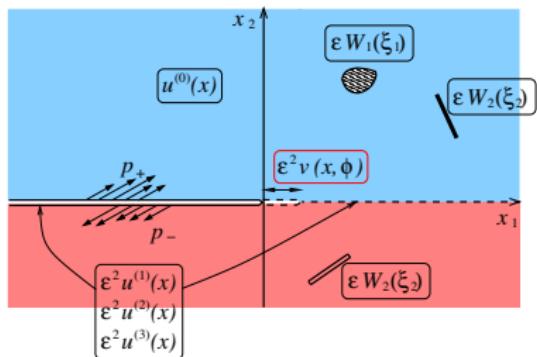
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- ➊ Solution of the unperturbed problem ($\varepsilon = 0$)
- ➋ Boundary layers concentrated near the defects
- ➌ Additional terms to adjust the BC and IC disturbed by the boundary layers

The asymptotic ansatz

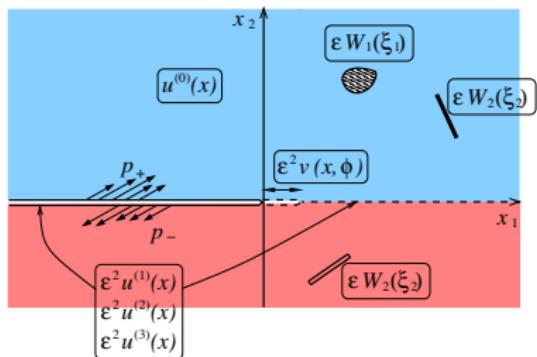
$$u(\mathbf{x}, \varepsilon) = u^{(0)}(\mathbf{x}) + \varepsilon \sum_{j=1}^3 W_j(\xi_j) + \varepsilon^2 \sum_{j=1}^3 u^{(j)}(\mathbf{x}) + \boxed{\varepsilon^2 v(\mathbf{x}, \phi)} + o(\varepsilon^2), \quad \varepsilon \rightarrow 0$$



- ➊ Solution of the unperturbed problem ($\varepsilon = 0$)
- ➋ Boundary layers concentrated near the defects
- ➌ Additional terms to adjust the BC and IC disturbed by the boundary layers
- ➍ Perturbation associated with the crack advance $\varepsilon^2 \phi$

The asymptotic ansatz

$$u(\mathbf{x}, \varepsilon) = u^{(0)}(\mathbf{x}) + \varepsilon \sum_{j=1}^3 W_j(\xi_j) + \varepsilon^2 \sum_{j=1}^3 u^{(j)}(\mathbf{x}) + \varepsilon^2 v(\mathbf{x}, \phi) + o(\varepsilon^2), \quad \varepsilon \rightarrow 0$$



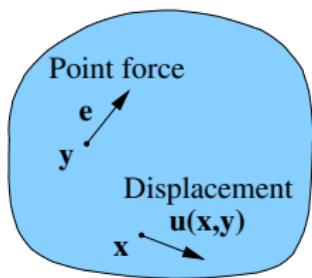
- ➊ Solution of the unperturbed problem ($\varepsilon = 0$)
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- Using the linearity of the problem, we analyse the perturbation of each defect separately (**superposition principle**).
- The method can be extended to a finite number of defects, provided that the distance between defects remains finite (**composite is dilute**).

The first challenge: Weight functions for an interfacial crack

"Weight functions are fundamental solutions for the problem of a cracked body"

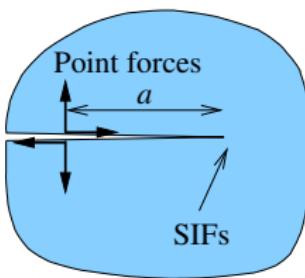
Linear elasticity:



Green's function
for elastic body

$$u_i(\mathbf{x}, \mathbf{y}) = G_{ij}(\mathbf{x}, \mathbf{y})e_j$$

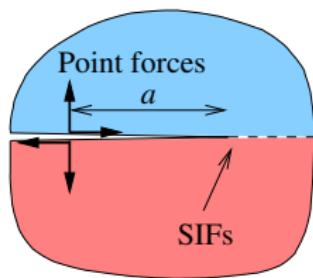
Linear fracture mechanics:



Weight functions for
cracked homogeneous body

$$K = \sqrt{\frac{2}{\pi}}(1+i)a^{-1/2}$$

Interfacial crack:



Weight functions for
interfacial crack

?

The weight functions for an interfacial crack

- Mode III symmetric and skew-symmetric weight functions:

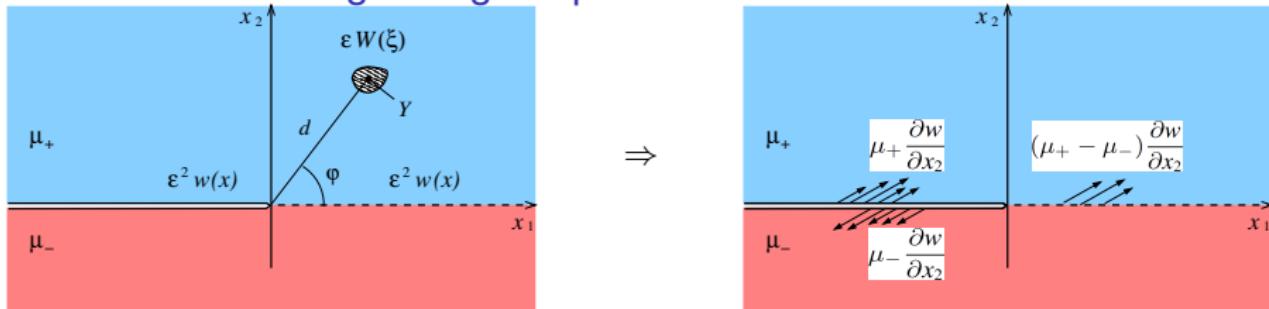
$$[\![U_3]\!](x_1) = \begin{cases} \frac{1-i}{\sqrt{2\pi}} x_1^{-1/2}, & \text{for } x_1 > 0, \\ 0, & \text{for } x_1 < 0, \end{cases} \quad \langle U_3 \rangle = \frac{\eta}{2} [\![U_3]\!],$$

- Integral formula for the computation of the Mode III SIF:

$$K_{\text{III}} = -(1+i) \lim_{x'_1 \rightarrow 0^+} \int_{-\infty}^0 \left\{ \underbrace{[\![U_3]\!](x'_1 - x_1) \langle p_3 \rangle(x_1)}_{\text{symmetric part}} + \underbrace{\langle U_3 \rangle(x'_1 - x_1) [\![p_3]\!](x_1)}_{\text{skew-symmetric part}} \right\} dx_1$$

$$\eta = \frac{\mu_- - \mu_+}{\mu_- + \mu_+} \quad \text{bimaterial parameter}$$

The second challenge: singular perturbation for a small defect



The defect is replaced by “effective” tractions

Dipole field: $\varepsilon^2 w(\mathbf{x}) = -\frac{\varepsilon^2}{2\pi} \left[\nabla u^{(0)} \Big|_{\mathbf{Y}} \right] \cdot \left[\mathcal{M} \frac{\mathbf{x} - \mathbf{Y}}{|\mathbf{x} - \mathbf{Y}|^2} \right] + o(\varepsilon^2), \quad \varepsilon \rightarrow 0$

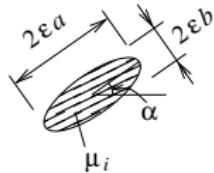
\mathcal{M} is the dipole matrix

Perturbation of the SIF:

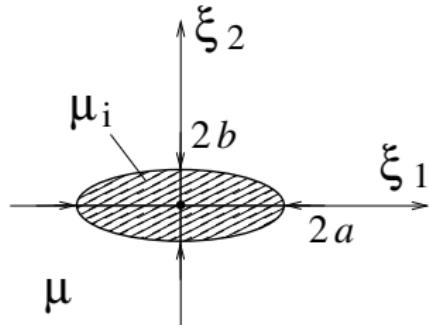
$$\Delta K_{\text{III}} = -\sqrt{\frac{2}{\pi}} \frac{\mu_+ \mu_-}{\mu_+ + \mu_-} \nabla u^{(0)} \Big|_{\mathbf{Y}} \cdot \mathcal{M} \mathbf{c}$$

$$\mathbf{c} = \frac{1}{2d^{3/2}} \left[-\sin \frac{3\varphi}{2}, \cos \frac{3\varphi}{2} \right]$$

Example: Elliptic elastic inclusion



$$\Rightarrow \quad \xi = \frac{\mathbf{x} - \mathbf{Y}}{\varepsilon} \quad \Rightarrow$$



Dipole field:

$$\varepsilon^2 w(\mathbf{x}) = -\frac{\varepsilon^2}{2\pi} \left[\nabla u^{(0)} \Big|_{\mathbf{Y}} \right] \cdot \left[\mathcal{M} \frac{\mathbf{x} - \mathbf{Y}}{|\mathbf{x} - \mathbf{Y}|^2} \right] + o(\varepsilon^2), \quad \varepsilon \rightarrow 0$$

Dipole matrix

$$\mathcal{M} = -\frac{\pi}{2} ab(1+e)(\mu_* - 1) \begin{bmatrix} \frac{1 + \cos 2\alpha}{e + \mu_*} + \frac{1 - \cos 2\alpha}{1 + e\mu_*} & -\frac{(1-e)(\mu_* - 1)\sin 2\alpha}{(e + \mu_*)(1 + e\mu_*)} \\ -\frac{(1-e)(\mu_* - 1)\sin 2\alpha}{(e + \mu_*)(1 + e\mu_*)} & \frac{1 - \cos 2\alpha_1}{e + \mu_*} + \frac{1 + \cos 2\alpha}{1 + e\mu_*} \end{bmatrix}$$

$$\mu_* = \mu/\mu_i \quad e = b/a$$

Dipole matrix for different types of defects



microcrack

$$\mathcal{M} = -\pi l^2 \begin{bmatrix} \sin^2 \alpha & -\sin \alpha \cos \alpha \\ -\sin \alpha \cos \alpha & \cos^2 \alpha \end{bmatrix}$$



rigid line inclusion

$$\mathcal{M} = \pi l^2 \begin{bmatrix} \cos^2 \alpha & \sin \alpha \cos \alpha \\ \sin \alpha \cos \alpha & \sin^2 \alpha \end{bmatrix}$$



elliptic void

$$\mathcal{M} = -\frac{\pi}{2} ab(1/e + 1) \begin{bmatrix} 1 - \cos 2\alpha + e(1 + \cos 2\alpha) & -(1 - e) \sin 2\alpha \\ -(1 - e) \sin 2\alpha & 1 + \cos 2\alpha + e(1 - \cos 2\alpha) \end{bmatrix}$$



elliptic rigid inclusion

$$\mathcal{M} = \frac{\pi}{2} ab(1/e + 1) \begin{bmatrix} 1 + \cos 2\alpha + e(1 - \cos 2\alpha) & (1 - e) \sin 2\alpha \\ (1 - e) \sin 2\alpha & 1 - \cos 2\alpha + e(1 + \cos 2\alpha) \end{bmatrix}$$



line defect (soft)

$$\mathcal{M} = -\frac{\pi}{2} \frac{l^2}{\kappa l + 1} \begin{bmatrix} 1 - \cos 2\alpha & -\sin 2\alpha \\ -\sin 2\alpha & 1 + \cos 2\alpha \end{bmatrix}$$



line defect (stiff)

$$\mathcal{M} = \frac{\pi}{2} \frac{\kappa l^2}{\kappa + l} \begin{bmatrix} 1 + \cos 2\alpha & \sin 2\alpha \\ \sin 2\alpha & 1 - \cos 2\alpha \end{bmatrix}$$

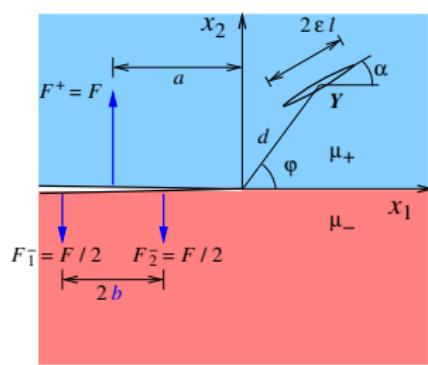
Application: Shielding and amplification effects

$$\Delta K_{III} = -\sqrt{\frac{2}{\pi}} \frac{\mu_+ + \mu_-}{\mu_+ - \mu_-} \nabla u^{(0)} \Big|_{\mathbf{Y}} \cdot \mathbf{M} \mathbf{c}$$

Definition:

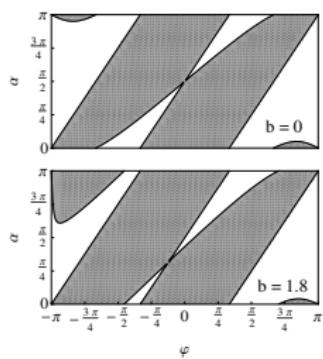
- | | | |
|------------------------|-----------------------------|--|
| $\Delta K_{III} < 0$: | <i>shielding effect</i> | "the defect is preventing the propagation" |
| $\Delta K_{III} > 0$: | <i>amplification effect</i> | "the defect is promoting the propagation" |
| $\Delta K_{III} = 0$: | <i>neutral</i> | "the defect has no effect" |

Example: shielding/amplification diagrams for macro-microcrack interaction



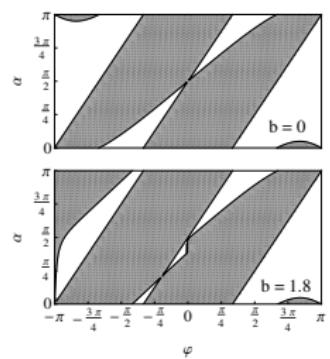
$$d = 1, 2\varepsilon l = 0.02, a = 2$$

Homogeneous plane:



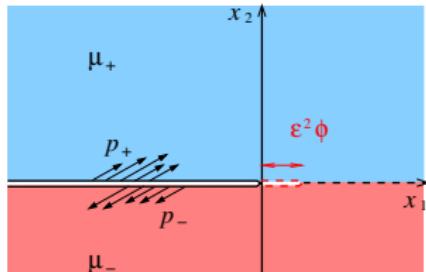
$$\mu_+ = \mu_-$$

Bimaterial plane:



$$\mu_+/\mu_- = 0.1$$

The third challenge: Singular perturbation for crack advance



The Betti identity:

$$\int_{-\infty}^{\infty} \left\{ [\![U]\!](x'_1 - x_1) \langle \sigma \rangle(x_1) + \langle U \rangle(x'_1 - x_1) [\!\sigma\!](x_1) - \langle \Sigma \rangle(x'_1 - x_1) [\!u\!](x_1) \right\} dx_1 = 0$$

u, σ physical solution

U, Σ weight functions

Perturbation of the SIF:

$$\varepsilon^2 \Delta K_{\text{III}}^\phi = \frac{\varepsilon^2 \phi}{2} A_{\text{III}}^{(0)}$$

$$A_{\text{III}}^{(0)} = \sqrt{\frac{2}{\pi}} \int_{-\infty}^0 \left(\langle p \rangle(x_1) + \frac{\eta}{2} [\![p]\!](x_1) \right) (-x_1)^{-3/2} dx_1$$

Analysis of a stable quasi-static propagation

The stress intensity factor is expanded as follows:

$$K_{\text{III}} = K_{\text{III}}^{(0)} + \varepsilon^2 \left(\Delta K_{\text{III}}^\phi + \sum_{j=1}^3 \Delta K_{\text{III}}^{(j)} \right) + o(\varepsilon^2), \quad \varepsilon \rightarrow 0$$

$\Delta K_{\text{III}}^\phi = \frac{\varepsilon^2 \phi}{2} A_{\text{III}}^{(0)}$: perturbation produced by the elongation of the crack along the interface

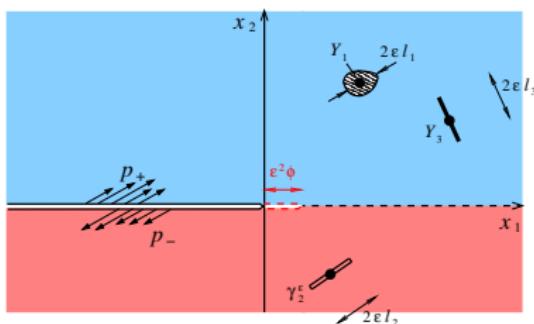
$\sum_{j=1}^3 \Delta K_{\text{III}}^{(j)}$: perturbation produced by the defects

We assume that the crack propagation is stable and quasi-static: $G = G_c$

$$G = \frac{1}{4} \left(\frac{1}{\mu_+} + \frac{1}{\mu_-} \right) K_{\text{III}}^2 \Rightarrow \Delta K_{\text{III}}^\phi + \sum_{j=1}^3 \Delta K_{\text{III}}^{(j)} = 0 \Rightarrow \phi = \frac{2}{A_{\text{III}}^{(0)}} \sum_{j=1}^3 \Delta K_{\text{III}}^{(j)}$$

$$\Delta K_{\text{III}}^{(j)} = -\sqrt{\frac{2}{\pi}} \frac{\mu_+ + \mu_-}{\mu_+ + \mu_-} \nabla u^{(0)} \Big|_{Y_j} \cdot \mathcal{M}_j \mathbf{c}_j, \quad \mathbf{c}_j = \frac{1}{2d_j^{3/2}} \left[-\sin \frac{3\varphi_j}{2}, \cos \frac{3\varphi_j}{2} \right]$$

Application: Crack propagation and arrest



Given a configuration of defects and position of the crack tip, the **incremental crack advance** ϕ is given by:

$$\phi = \frac{2}{A_{\text{III}}^{(0)}} \sum_{j=1}^3 \Delta K_{\text{III}}^{(j)}$$

It is possible to update the configuration with the new position of the crack tip and recompute the incremental crack advance in the new configuration, following an iterative procedure:

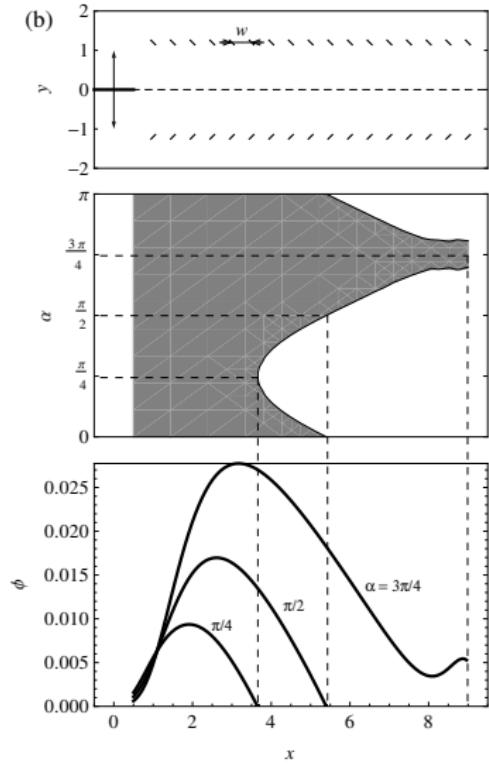
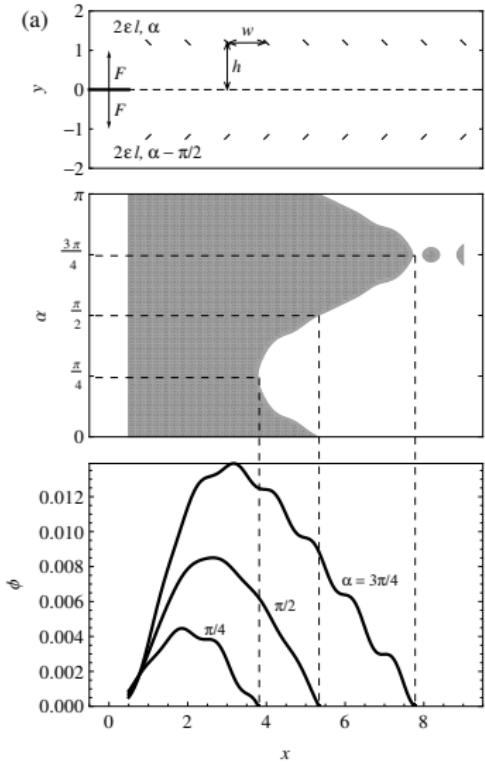
- the crack “accelerates” when the increment ϕ is increasing
- the crack “decelerates” when the increment ϕ is decreasing
- the crack “arrests” when a neutral configuration is reached ($\phi = 0$)

The **total crack elongation** is computed as:

$$x(N) = \sum_{i=0}^N \epsilon^2 \phi_i$$

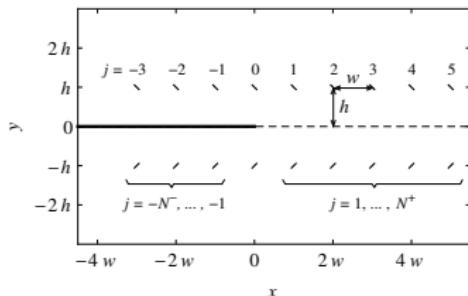
where N is the number of iterations.

Crack propagation in a finite array of defects



Conclusions and possible extensions

This asymptotic method can be extended to the case of an infinite array of defects



$$\phi = \frac{al^2 \cos 2\alpha}{2h^2} \left\{ -1 + \sum_{j=1}^{N^+} \frac{\left(1 - \frac{h^2}{j^2 w^2}\right) \frac{h^2}{j^2 w^2}}{\left(1 + \frac{h^2}{j^2 w^2}\right)^2} + \sum_{j=1}^{N^-} \frac{\left(1 - \frac{h^2}{j^2 w^2}\right) \frac{h^2}{j^2 w^2}}{\left(1 + \frac{h^2}{j^2 w^2}\right)^2} \right\}$$

Take the limit as $N^+ \rightarrow \infty$:

$$\phi = \frac{al^2 \cos 2\alpha}{2h^2} \left\{ -\frac{1}{2} - \frac{\left(\frac{\pi h}{w}\right)^2}{2 \sinh^2\left(\frac{\pi h}{w}\right)} + \sum_{j=1}^{N^-} \frac{\left(1 - \frac{h^2}{j^2 w^2}\right) \frac{h^2}{j^2 w^2}}{\left(1 + \frac{h^2}{j^2 w^2}\right)^2} \right\}$$

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