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Research Paper

George William Scott Blair – the pioneer of fractional calculus in rheology

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S.R. dedicates this article to his friend and co-author
Professor Francesco Mainardi
on the occasion of his retirement

Communicated by Gianni Pagnini and Enrico Scalas

Abstract

The article shows the pioneering role of the British scientist, Professor G.W.Scott Blair, in the creation of the application of fractional modelling in rheology. Discussion of his results is presented. His approach is highly recognized by the rheological society and is adopted and generalized by his successors. Further development of this branch of Science is briefly described in this article too.

Keywords: fractional calculus, rheology, Scott Blair's fractional model, hereditary solid mechanics, relaxation.

AMS subject classification: 26A33, 60G22, 74-03, 74D05, 76-03, 76A10.

1. Introduction.

The interest to applications of fractional calculus in modelling of different phenomena in Physics, Chemistry, Biology is rapidly increasing in the recent three decades. First of all we have to point out the constitutive modelling of non-Newtonian fluids. The main reason is that the fractional models give us possibility for simple description of complex behaviour of a

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viscoelastic material. In pioneering (mainly experimental) works of 1940-1950th, it was discovered, for instance, that the relaxation processes in some materials exhibit an algebraic decay, which cannot be described in the framework of the Maxwell model based on exponential behaviour of the relaxation moduli.

In order to see perspective in the development of fractional models it is important to understand how such models appear, and what was really done by the pioneers. Among the works which are in the core of the first period of the fractional modelling one can single out the series of articles and monographs by G.W. Scott Blair. His role is not overestimed by the rheological society (see, e.g. [1], [2]), but anyway, some details of his work are still of great importance. We propose here an analysis of the results by G.W. Scott Blair along with their influence on the modern development of the fractional modelling in rheology.

1.1. *Short biography by G.W. Scott Blair*

Dr. George William Scott Blair (1902–1987) was born on 23 July 1902, of Scottish ancestry, in Weybridgein Surrey, England. After graduated a famous public school at Charterhouse he went to Trinity College Oxford in 1920, where he studied Chemistry, with Prof. Sir Cyril Hinshelwood as his tutor. He carried out a one-year research project in colloid chemistry to complete his master thesis with honour degree.

After graduating Scott Blair was employed as a colloid chemist with a Manchester firm of Henry Simon, working there on the viscometry of flour suspensions, publishing his first rheology paper in 1927. In 1926 he was offered a post in the Physics Department of the Rothamsted Experimental Station, where he was working on the flow properties of soils and clays until 1937. It was there, where he made with his colleagues the first quantitative study of so called sigma-phenomenon, which was originally described by Bingham and Green in 1919. Schofield and Scott Blair (see [3]) studied this phenomenon from 1930 at Rothamsted for soil and clay pastes and named it "sigma effect". These studies were probably unknown to Fåhraeus and Lidquist, who first discovered the sigma effect for blood, referred to as the "Fåhraeus-Lindquist phenomenon". At this period some preliminary experiments were provided by Scott Blair which led him later to the necessity to consider anomalous relationship between stress, strain and time (see, e.g., [4]).

In 1929 Scott Blair took up a Rockefeller Fellowship at Cornell University in Ithica, New York state. He worked there on the flow of potter's clay and developed a means of measuring its plasticity. He attended the inaugu-

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ral meeting of Society of Rheology in December 1929 in Washington DC, met there all pioneers of Rheology, Eugune Bingham and Markus Reiner among them, and many became his life-time friends.

After returning to Rothamsted, Scott Blair made rheological research on honey and flour doughs. He also studied together with the well-known psychologist David Katz, psychophysical problems in bread making. His interest in psychology led him, together with F.M.V. Coppen, to initiate a new field, for which he coined the word "psychorheology". It is considered as one of the fields of biorheology.

In 1936 he submitted his PhD thesis to the University of London, and it was examined by Prof. Freundlich. The same institution later awarded him a D.Sc. for his labours in rheology (probably the first ever rheology D.Sc.) In 1937 he joined National Institute for Research in Dairying, University of Reading as a head of Chemistry but soon took over the newly formed Physics Department and remained in that position until his retirement thirty years later.

In 1940 the British Rheological Society was founded. Scott Blair played a prominent role and took active part in the development of rheology. He was a Founder-Member and, later, president of the British Society of Rheology (1949–1951). He took part in the organization of the First International Congress on Rheology, held at Scheveningen, Holland, in 1948. He was a Secretary of the Second International Congress on Rheology in Oxford in 1953 and a member of Committee on Rheology, set up by the International Council of Scientific Union. Scott Blair was given to flights of fancy into psychorheology, fractional differentiation etc. In 1969 he was awarded the Poiseuille Gold Medal of the International Society of Haemorheology (now Biorheology) and in 1970 he received the Founders Gold Medal of the British Society of Rheology. Together with J. Burgers he published a monograph on rheological nomenclature [5]. For many years Scott Blair was the Chairman of the British Standard Institute Committee on Rheological Nomenclature.

During almost a half of century George W. Scott Blair was one of the leading rheologists. Beginning from 1957 Scott Blair devoted his experimental and theoretical work entirely to hemorheology. Since the foundation of the International Society of Hemorheology in Reykjavik, Iceland in 1966, he was a member of its Council and acted as Chairman of its Committee on Standards and Terminology. After he retired he worked on the flow and coagulation of blood at the Oxford Haemophilia Centre.

Scott Blair was very active in publication and editorial work. He was a co-founder of the Journal "Biorheology" and its Co-Editor-in-Chief from its inception in November 1959 to December 1978 (see [6]). The books and research papers of Scott Blair were donated to the British Society of Rheology and later deposited in the Library of Aberystwyth University in early 1980's. The collection has over 550 books and its aim is to develop this "into an up-to-date library of rheological literature available to all members of Society". Rheology Abstracts and the British Society of Rheology Bulletin are two journals published by/for the Society which form an important of the Collection. The books and journals catalogued online (access via http://primo.aber.ac.uk).

2. Rheology and Psychophysics

It was Professor Bingham who had chosen the name "Rheology" for this branch of the Science and gave the definition of it: "The Science of Deformation and Flow of Matter" (see [7]) motivated by Heraclitus' quote "$\pi\alpha\nu\tau\alpha\ \rho\varepsilon\iota$" ("everything flows"). Rheology is one of the very few disciplines having exact day of its birth, April 29, 1929, when the preliminary scope of the Society of Rheology was set up by a committee met at Columbus, Ohio. Anyway, the ancient Egyptian scientist Amenemhet (ca. 1600 BC), who made the earliest application of the viscosity effect, can be considered as the first rheologist (see, *e.g.*, [8]).

The observables in rheology are deformations or strains, and the changes of strains in time. Changes of strains in time constitute a flow. Thus, these changes are generally associated with internal flow of certain kind. States of stress are inferred either from the comparative strain behaviour of complex and simple systems in interaction or from the behaviour of a known mass in the gravitational field. In physical testing, stresses (S), strains (σ) or their differentials with respect to time $\left(\dot{S}, \dot{\sigma}\right)$ are normally held constant, leaving either a length to be measured, or the time (t).

There is a group of fluids which is characterized by a coefficient of viscosity for a specific temperature. These fluids, known as Newtonian fluids, were singled out by Newton who proposed the definition of resistance (or viscosity in modern language) of an ideal fluid. Pioneering work on the laws of motion for real (i.e. non-ideal) fluids with finite viscosities was carried out by Navier [9] and later by Stokes [10]. The Navier-Stokes equation enabled, among other things, prediction of velocity distributions and flow between rotating cylinders and cylindrical tubes (see [2]).

Nowadays rheology generally accounts for the behaviour of non-Newtonian fluids, by characterizing the minimum number of functions that are needed to relate stresses with rate of change of strains or strain rates. This kind of fluids is called Newtonian since Newton's introduction the concept of viscosity.

In practice, rheology is concerned with extending continuum mechanics to characterize flow of materials, that exhibits a combination of elastic, viscous and plastic behaviour by properly combining elasticity and (Newtonian) fluid mechanics. In [2] the main directions in the development of the rheology are described. First of all this is linear *viscoelasticity*. One of the most important early contribution in this area is the work by Maxwell [11]. In order to explain the behaviour of the materials which are neither truly elastic nor viscous he proposed a constant relaxation time (t_r) and justified implicitly the model of a dash-pot and spring in series. Anyway, he realized that for some materials the assumption of constancy of the relaxation time is over-simplification, in these cases t_r has to be a function of stress. Meanwhile, the notion of Maxwell's units (i.e. pieces of a material having constant relaxation time) has been widely explored by rheologists. Later the conception of the "orientation times" τ has been developed (see, *e.g.*, [12]). It is considered unit stress conditions, supposing that the strain is approaching to an equilibrium value. Thus, the immediate Hookean strain is first subtracted and τ is defined as the time taken for the remaining strain, resulting from the orientation of the chains.

Another direction which was singled out in [2] is the study of *generalized Newtonian materials*. This type of fluid behaviour is associated with the work by Bingham [13] who proposed so called yield stress to describe the flow of paints. In [14], it has been pointed out the close similarity between the usual experimental Bingham curve and the curve of a high power-law. Thus, it shows possibility of existence of systems for which the Bingham plot gives a fairer and simpler account of the data.

The study of *non-linear viscoelasticity* started at the begining of XXs century, when the area of rheology was most rapidly grown (see, *e.g.* [2]). Thus, Poynting [15] in his experiment discovered that loaded wires increased by a length that was proportional to the square of the twist, what did not correspond to the usual expectation of the linear viscoelasticity theory. Probably the first theoretical work on non-linear viscoelasticity was done by Zaremba [16], who extended linear theory to the non-linear regime by introducing corotational derivative in order to incorporate a frame of reference that was translating and rotating with the material. More extended description of the results in non-linear viscoelasticity can be found in [2] (see also [17], [18] and references therein).

Not all properties of flowing matter can be interpreted in term of real rheological sense. In this case psychophysical approach with its psychophysical experiments can be helpful. Psychophysics is defined as the scientific study of the relation between stimulus and sensation (see, *e.g.* [19]). Psychophysicists usually employ experimental stimuli that can be objectively measured. Psychophysical experiments have traditionally used three methods for testing subjects' perception in stimulus detection and difference detection experiments: the method of limits, the method of constant stimuli and the method of adjustment.

G.W. Scott Blair widely used psychophysical experiments in his research (see [20]). Therefore, it is interesting to recall how he described the role of psychophysics in rheology (see [7]): "The complex and commercially important rheological "properties" of many industrial materials are still assessed subjectively by handling in factory and are expressed in terms "body", "firmness", "spring", "deadness", shortness", "nerve", etc. - concepts which cannot be interpreted ... in terms of simple rheological properties at all. In view of this fact ... it is clearly advisable to know something of the accuracy with which these handling judgements can be made and, by bulking sufficiently large numbers of data together so that reproducibility is ensured, to attempt to correlate the entities so derived with manageable functions of $S : \sigma : t$. A start has been made in this direction and not only have a number of reproducible regularities been observed, but the information obtained has laid the foundations of a theory of "*Quasi-properties*" which it is hoped will facilitate the study of purely "physical" rheology of complex materials."

This observation is a core of Scott Blair's method which he used along his career.

3. Scott Blair Fractional Element

3.1. *Nutting's Law*

In 1921 Nutting reported (see [21]) about his observation that mechanical strains appeared at the deformation of the viscoelastic materials decreasing as power-type functions in time. From a series of experiments, which covered a range of materials from the elastic solid to the viscous fluid, Nutting suggested a general formula relating shear stress, shear strain and time, whenever shear stress remains constant:

$$\sigma(t) \sim C\Delta S \cdot t^{-\alpha}, \tag{1}$$

with constant order $\alpha \in (0, 1)$ which is close to $1/2$ for many materials (see also his more later work [22], [23]). This conclusion was in a strong contradiction to the standard exponential law. Later the Nutting's observation was justified by Gemant who studied the properties of viscoelastic materials under harmonic load. It was shown that the memory function $\eta(t)$ can have power-type relaxation behaviour proportional to $t^{-3/2}$. In 1950 Gemant published a series of 16 articles entitled "Frictional Phenomena" in Journal of Applied Physics since 1941 to 1943, which were collected in a book of the same title [24]. In his eighth chapter-paper [25, p. 220], he referred to his previous articles [26], [27] for justifying the necessity of fractional differential operators to compute the shape of relaxation curves for some elasto-viscous fluids.

Gemant has used half-differential, but is his later papers he says that fractional differential "only occurs as a useful mathematical symbol, whereas the underlying elementary process, whatever it may be, will probably contain differential quotient of an integral order".

Scott Blair surely knew the attempts by Gemant (see, *e.g.*, [28]) to generalize Maxwell's theory by changing various (but integer) powers in complex modulus of the Maxwell Fluid Model to fractional powers. In fact, Scott Blair (together with Coppen) also came to the form of Nutting equation, but from another consideration. They argued that the material properties are determined by various states between an elastic solid and a viscous fluid, rather than a combination of an elastic and a viscous element as proposed by Maxwell. In [29] it was pointed out that, since for Hookian solids, strain is proportional to stress and to unit power of time, for intermediate materials, it might be expected to be proportional to stress and to some fractional power of time with exponent $\alpha, 0 < \alpha < 1$ and described this relation in the form

$$\psi = S^{\beta}\sigma^{-1}t^{\alpha}, \tag{2}$$

where proportionality coefficient ψ is a constant. Derived in this way the equation (2) looks entirely empirical, tough the fundamental significance of α (which is called the *dissipation coefficient*) is shown in psychophysical experiments described by Scott Blair and Coppen (see [30], [31], [32] and [33]).

A comparison of the partially differentiated Nutting equation and Maxwell's equation may be written (see [34]), namely, for Nutting:[a]

$$-\left(\frac{\partial S}{\partial t}\right)_{\sigma} = \frac{\alpha}{\beta}\frac{S}{t}, \tag{3}$$

and for Maxwell:

$$-\left(\frac{\partial S}{\partial t}\right)_{\sigma} = \frac{S}{t_r}. \tag{4}$$

Since Nutting equation gives $t = \psi^{1/\alpha}S^{-\beta/\alpha}\sigma^{1/\alpha}$, it is apparent that the Nutting treatment postulates a single *relaxation time* proportional to a

[a] The suffix σ indicates shear strains.

power of the stress. This is simplest possible way of implementing Maxwell's suggestion that relaxation time t_r may be some function of stress. From the other side it justifies the believe that the use of fractional calculus in description of processes toward equilibrium is necessary if one has to keep the Newtonian time scale.

3.2. *Scott Blair's fractional model*

It was suggested in [32] that, in considered cases, comparative firmness is judged neither by σ, nor by $\dot{\sigma}$, nor by any mixture of these two magnitude, but by some intermediate entity, namely by fractional derivative.[b] More exactly, he wrote: The general constitutive equation "... is applicable to integral values of n but a more general equation may be used even n is a fraction.The numerical coefficient is expressed as a quotient of Γ-functions and may be written

$$\frac{\delta^n \sigma}{\delta t^n} = \frac{\Gamma(k+1)}{\Gamma(k-n+1)} t^{k-n} \Psi^{-1} S. \tag{5}$$

The expression $\Gamma(k+1)$ is given by $\int\limits_0^\infty e^{-x} x^k \, dx$ ...". This model was reported also by Scott Blair in Nature [35].

In his work Scott Blair did not specify what kind of fractional derivative he used. From the way how he has calculated derivative of any power we can conclude that this is the standard Riemann-Liouville derivative of non integer-order. In fact for this derivative in modern notation we have

$$D^{\mu}_{0+} t^{\alpha} = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+1-\mu)} \, t^{\alpha-\mu} \quad \alpha > -1 \, , \quad \mu \geq 0 \, , \tag{6}$$

with the correspondence $n \to \mu$ and $k \to \alpha$. In practice the fractional derivative of the power law was already used by Euler.

It is quite instructive to cite some words by Scott-Blair quoted by Stiassnie in their correspondence, see [36]: *I was working on the assessing of firmness of various materials (e.g. cheese and clay by experts handling them) these systems are of course both elastic and viscous but I felt sure that judgements were made not on an addition of elastic and viscous parts but on something in between the two so I introduced fractional differentials of strain with respect to time.* Later, in the same letter Scott-Blair added:

[b] In fact, few misprints had been made in [32] later corrected by Scott Blair. Thus, in the original reprint of [32] in the Scott Blair reading room one can find hand-written corrections made by Scott Blair.

I gave up the work eventually, mainly because I could not find a definition of a fractional differential that would satisfy the mathematicians.

The above said *Principle of Intermediacy* was discussed in details by Scott Blair in [34] basing on purely physical grounds. The theory of fractional modelling in rheology is developed by Scott Blair, Veinoglou and Caffyn in [28]. In [7, p. 30] it is briefly summarized: "... times are normally defined as equal when "free" Newtonian bodies (or alternatively light) traverse equal (superposable) distances in them. This leads to a a definition of velocity as the first differential of length with respect to time which, because of this definition of time equality, is constant for Newtonian bodies; and to the second differential, called acceleration.

When bodies are not influenced by other bodies, and their velocities change with time, a force is postulated and defined as rate of change of (velocity $\times$ mass). It is long been realized that the Newtonian time scale arbitrary (see [37, p. 80]) and in the case of a complex plastic being strained, the rheologically active units are certainly not independent Newtonian bodies. It should, therefore, be easy to choose a non-Newtonian time equality definition which would reduce the entities by which firmness is judged to simple whole-number[c] differential expression. The use of separate time scales for different materials is not convenient, however, so Newtonian time is used, but, as a result of this arbitrary procedure, the derived constants cannot be expected to be built up entirely from whole-number differentials. It is thus apparent that fractional differential is an essential feature of our whole mode of approach."

In [38] are discussed the circumstances under which it is practicable to express the Nutting equation and its fractional derivatives in a simple dimensional form. Three main principles of a new proposal are formulated: (1) the fact that the treatment does not lead to any understanding of structure of the materials or of their molecular configurations; (2) the only entities are used whose dimensions depend of the nature of of the material; (3) fractional derivatives and corresponding coefficients are understood as something intermediate between zero and first derivatives and corresponding coefficients. Scott Blair highly supported (see, it e.g. [14]) the ideas by Nutting supposing that for that moment it describe a special but very frequently adequate cases. Anyway, he though that the phenomenon dealing with Nutting equation are related to the fundamental structure of materials.

Fractional derivative of order $\mu, 0 < \mu < 1$, with respect to time t of the

[c] I.e. non-integer order.

Nutting equation (in the form (2)) gives, using notation (6) (see [32], [28])

$$\frac{\partial^\mu \sigma}{\partial t^\mu} = D^\mu_{0+} t^\alpha \psi^{-1} S. \tag{7}$$

Another way to justify this relation is to introduce quasi-property χ_1 by the *Principle of Intermediacy*

$$\chi_1 = S \div \frac{\partial^\mu \sigma}{\partial t^\mu}, \tag{8}$$

since the viscosity can be defined by the relation $\eta = S \div \frac{d\sigma}{dt}$, and shear modulus as $n = S \div \sigma$.

The relation (8) can be integrated to give

$$\sigma = S^\beta \left(A t^{\alpha'} + B t^{\alpha'-1} + C t^{\alpha'-2} + \dots \right), \tag{9}$$

where A, B, C etc. are constants. Clearly this equation coincides with Nutting equation if $A \gg B, C \dots$.

It should be noted that, speaking in modern language (see [39]), the Scott Blair equation is only relevant for special case of power-law creep functions. Another remark concerns the approach by Scott Blair. It was criticized by applied mathematicians since only few mathematical explanations were presented in his work (see discussion in Subsec. 3.2). Therefore from paper to paper he tried to make his idea more understandable. In particular, he changed his notation which makes certain difficulties to read Scott Blair's papers now.

4. Fractional models in rheology

After the first applications of the fractional derivatives in the modelling of the processes in rheology several other fractional models were proposed to describe certain rheological phenomenon. We briefly outline here the most discussed models of such a type.[d]

Gerasimov [40][e] used similar arguments as Scott Blair (as in [32], [28]), namely, interpolation between Hook and Newton's law, in order to introduce a rheological constitutive equation in terms of a precise notation of fractional derivative.

[d] Here and in what follows we will use modern notations for stress (σ) and strain (ε) that are not be confused with the corresponding notations used by Scott Blair, namely (S) for stress and (σ) for strain. Furthermore we will write D^α_{0+} to denote the Riemann-Liouville fractional derivative implicitly adopted by Scott Blair.

[e] We take liberty to describe the work by Gerasimov following the monograph by Uchaikin [41].

He used such equation (see below (10)) for description two models, namely, the flow of the viscoelastic between two parallel plates, and the rotational viscoelastic flow between two concentric cylinders. He obtained an exact solution by using operational method. Gerasimov has started his consideration by appealing to the Boltzmann equation. He said that from experiments follow the importance of a special case of the Boltzmann equation corresponding only to the hereditary part of the stress $\sigma(t)$. He pointed out the processes for which $\sigma(t)$ has a memory depending on the velocity of all earlier deformations, for which such equation has the form

$$\sigma(t) = \int_0^\infty K(\tau)\dot{\varepsilon}(t-\tau)d\tau. \tag{10}$$

For the kernel in this integro-differential relation (relaxation function) he claimed that for certain materials it can be written as

$$K(\tau) = \frac{A}{\tau^\alpha}, \quad A > 0, \quad 0 < \alpha < 1.$$

Hence, equation (10) can be written as

$$\sigma(t) = \frac{\kappa_\alpha}{\Gamma(1-\alpha)} \int_0^\infty \frac{\dot{\varepsilon}(t-\tau)}{\tau^\alpha} d\tau. \tag{11}$$

One can mention that in his representation of the model Gerasimov has used integration up to ∞. Later in [40] considering concrete mechanical problems he changed the interval of integration to $(0, t)$.

In [41, Ch. 12] it is discussed how it can be interpreted. If we consider $K(\tau)$ as reaction (response) at the time instant t of the stress on the step-like deformation of the unit value at the time instant $t - \tau$. Approximating the smooth function $\varepsilon(t)$ by step-like function with jumps $\Delta\varepsilon(t_j)$ at t_j and using linearity principle we get the integral sum

$$\sigma(t) = \sum_{t_j < t} K(t - t_j)\Delta\varepsilon(t_j),$$

and taken limit as $\Delta t_j \to 0$ we arrive at the integral form of the equation

$$\sigma(t) = \int_0^t K(t-t')d\varepsilon(t') = \int_0^t K(\tau)\dot{\varepsilon}(t-\tau)d\tau = A \int_0^t \frac{\dot{\varepsilon}(s)ds}{(t-s)^\alpha}. \tag{12}$$

It means that relaxation function $K(\tau)$ is defined only for $\tau > 0$.[f] In particular, equation (11) for $\alpha = 1$ gives us the Newton law, and for $\alpha = 0$ corresponds to Hookean law.

Similar to (10) formulation of the fractional model was proposed by Slonimsky [42]).

Rabotnov (see [43] and more extended description in his monograph [44]) presented a general theory of hereditary solid mechanics using integral equations (see also [45], where the use of integral equations for viscoelasticity was revisited and interjects fractional calculus into Rabotnov's theory by the introduction of the spring-pot was presented).

Rabotnov introduced an hereditary elastic rheological model with constitutive equation in form of Volterra integral equation with weakly singular kernel of special type[g]

$$\sigma(t) = E\left[\varepsilon(t) - \beta\int_0^{t_\alpha} R_\alpha(-\beta, t_\alpha - \tau)\varepsilon(\tau)d\tau\right], \tag{13}$$

where t_α is the aging time, $\alpha \in (-1, 0]$, $\beta \neq 0$, and the kernel R is represented in the form of power series

$$R_\alpha(\beta, x) = x^\alpha \sum_{n=0}^{\infty} \frac{\beta^n x^{n(\alpha+1)}}{\Gamma((n+1)(\alpha+1))}. \tag{14}$$

Rabotnov's kernel function $R_\alpha(\beta, x)$ is related to the well-known Mittag-Leffler function $E_{\alpha,\beta}(z)$ highly explored nowadays in the fractional calculus and its applications, namely

$$R_\alpha(\beta, x) = x^\alpha E_{\alpha+1,\alpha+1}(\beta x^{\alpha+1}). \tag{15}$$

In [46] the results by Rabotnov was summarized. It is said, that Rabotnov, in his book [44], presented a general theory of hereditary solid mechanics using integral equations. Koeller [45] (see also [39]) reviewed the use of integral equations for viscoelasticity and interjects fractional calculus into Rabotnov's theory by the introduction of the spring-pot, which he used to generalize the classical models. Meshkov et al. [47] as well as Rossikhin and Shitikova [48] (see also [49]) described and polularized Rabotnov's theory. In particular, they pointed out that Rabotnov's fractional exponential function is related to the well known Mittag-Leffler function and they showed

[f] Thus, up to the constant multiplier, the right-hand side of (12) coincides with the fractional derivative known as Caputo derivative.

[g] The integral stress - strain relationship by Rabotnov can be re-interpreted in terms of the fractional differential constitutive equation of Zener type, see later.

the equivalence of Rabotnov's theory to Torvik and Bagley's fractional polynomial constitutive equation. Further references on Rabotnov's theory may be found in [48], [49].

Of course Scott-Blair did not know the Mittag-Leffler function and its asymptotic behaviours (stretched exponential for short times and power law for long times). Presumably that Scott-Blair had guessed the behaviour of the M-L function but he did not have the mathematical background being overall an experimentalist.

Both Scott Blair's model (7) and Gerasimov's model (11) are naturally considered later as special cases of *fractional Maxwell's model* with rheological constitutive equation of the form

(16) $$\sigma(t) + \lambda^{\alpha} D^{\alpha}_{0+,t}\sigma(t) = E\lambda^{\beta} D^{\beta}_{0+,t}\varepsilon(t), \quad 0 \leq \alpha \leq \beta \leq 1,$$

where E is the shear modulus, and λ is the relaxation time. This equation generalizes celebrating Maxwell equation in which for the first time Newtonian law for viscous fluid and Hook's law for elastic solid are combined to describe the behaviour of visco-elastic media

(17) $$\sigma(t) + \tau D_t\sigma(t) = E\tau D_t\varepsilon(t).$$

Partial case of fractional Maxwell's model is the so-called *three-parametric generalized Maxwell's model* with constitutive equation of the type

(18) $$\sigma(t) + a_1 D^{\alpha}_{0+,t}\sigma(t) = b_0\varepsilon(t), \quad 0 < \alpha < 1.$$

Another popular fractional model with three parameters is the *Kelvin-Voigt fractional model* that presumably for the first time was introduced by Caputo [50] in 1967,

(19) $$\sigma(t) = b_0\varepsilon(t) + b_1 D^{\alpha}_{0+,t}\varepsilon(t), \quad 0 < \alpha < 1.$$

It is a generalization of the classical Kelvin model having the following constitutive equation

(20) $$\sigma(t) = E\left[\sigma(t) + \tau D_t\varepsilon(t)\right].$$

More general constitutive equation corresponds to the so called *fractional Zener model*:

(21) $$\sigma(t) + a_1 D^{\alpha}_{0+,t}\sigma(t) = b_0\varepsilon(t) + b_1 D^{\alpha}_{0+,t}\varepsilon(t), \quad 0 < \alpha < 1.$$

formerly introduced in 1971 by Caputo and Mainardi [51]. Theoretical background for this was done by Bagley and Torvik, see [52], [53]. It has to be pointed out that the above considered Rabotnov's model (13) is equivalent to the fractional Zener model, see [41].

Sometimes the *Poynting-Thomson fractional model* is discussed with rheological constitutive equation of the type

$$\sigma(t)+\frac{E}{E_0}\lambda^{\alpha-\gamma}D_{0+,t}^{\alpha-\gamma}\sigma(t)+\frac{E}{E_0}\lambda^{\beta-\gamma}D_{0+,t}^{\beta-\gamma}\sigma(t)=E\lambda^{\alpha}D_{0+,t}^{\alpha}\varepsilon(t)+E\lambda^{\beta}D_{0+,t}^{\beta}\varepsilon(t), \tag{22}$$

$(0 \leq \gamma \leq \alpha \leq \beta \leq 1.)$

More extended discussion of the fractional models in rheology can be found in [54], [17], [55], [56], [57].

5. The second generation of fractional modelling in rheology

The development of the fractional differential approach in rheology is associated with such names as Scott-Blair, Bagley and Torvik, Caputo, Gorenflo and Mainardi [58], Friedrich, Schiessel and Blumen [59], [56], Metzler, Nonnenmacher, Glöckle, Klafter and Schlesinger [60], [61], Koeller [46], Podlubny and Heymans [62], Rossikhin and Shitikova [47], [48], [49], and others.

In a series of papers (see, *e.g.*, [52], [53]) Bagley and Torvik extended the ideas of Gemant, and of Caputo–Mainardi [54]. They have shown that the complex modulus of many materials can be approximated by fractional powers in the frequency. They proposed a general fractional constitutive relation describing visco-elastic behaviour in its different appearance.

This approach has been successfully applied to describe rheological behaviour of organic glasses, elastomers, polyurethane, polyisobutylene, monodisperse polybutadiene and solid amorphous polymers in a wide temperature range (see for example [63] and references therein). In [64] and [65] have been derived equations governing the time-dependent indentation response for axisymmetric indenters into a fractional viscoelastic half-space and have proposed an original method for the inverse analysis of fractional viscoelastic properties and applied to experimental indentation creep data of polystyrene. The method is based on fitting the time-dependent indentation data (in the Laplace domain) to the fractional viscoelastic model response. It is shown that the particular time-dependent response of polystyrene is best captured by a bulk-and-deviator fractional viscoelastic model of the Zener type. We shall dwell in details on fractional differential models of viscoelasticity and then consider a few standard hydrodynamic problems in the simplest model of this type.

It is impossible to have a complete description of the modern state in the fractional rheology. We refer the interested readers to the recent

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monographs [17], [18], and to the survey paper [66] for some additional comments on pioneering works in applications of fractional calculus.

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