Numerical Modeling of Hydraulic Fractures: State of Art and New Results

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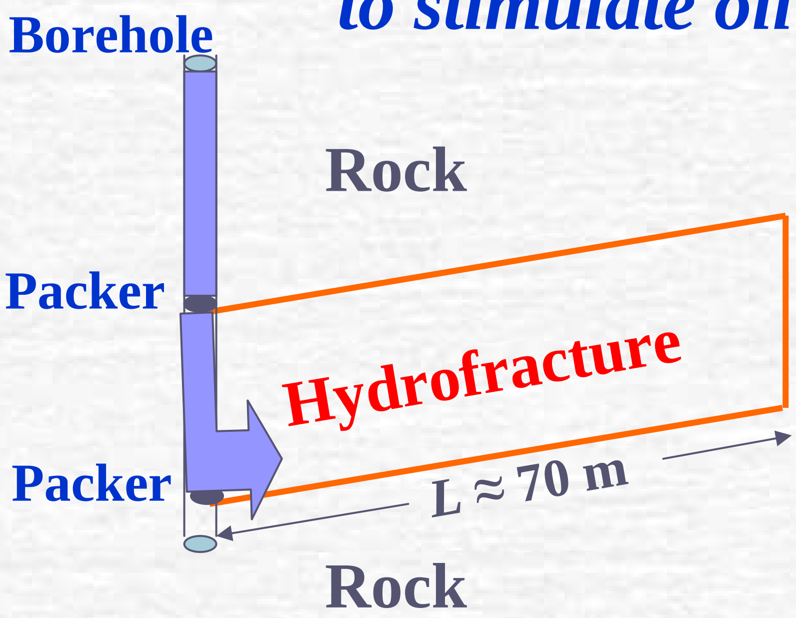
Hydraulic Fracturing

PART I

STATE OF ART

Hydraulic Fracturing
What is it?

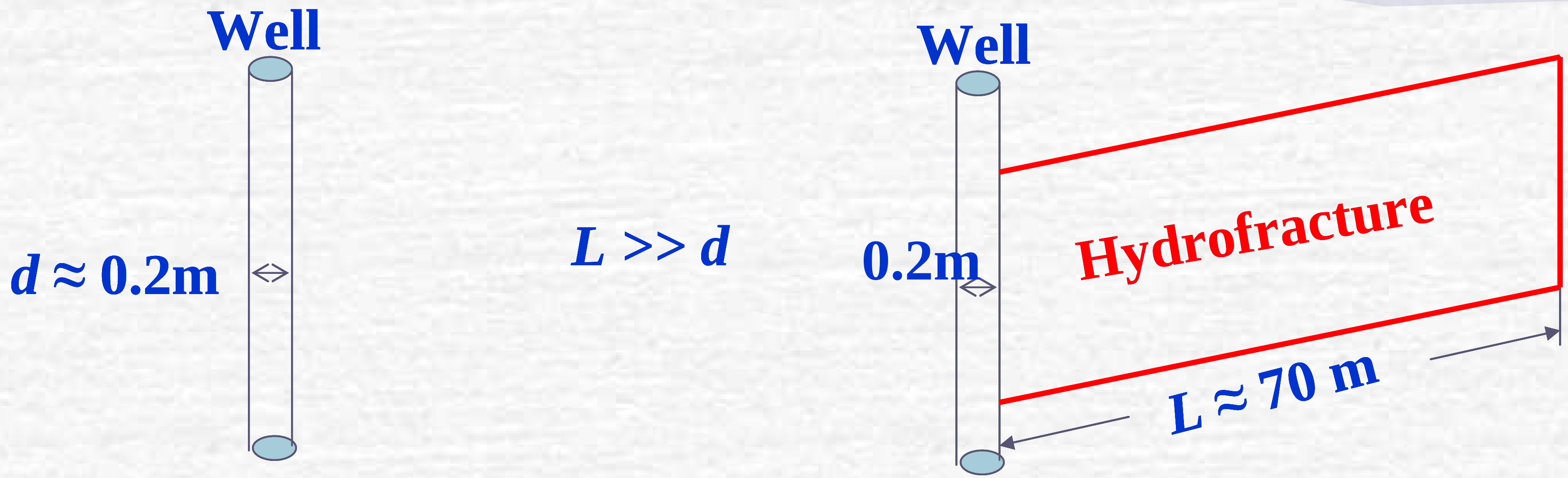
Hydraulic fracturing is the operation extensively used in the petroleum industry to stimulate oil and gas recovery



Water under high pressure is pumped between packers to create a crack (*hydrofracture*) in a productive layer

Thousands of treatments are successfully pumped each year

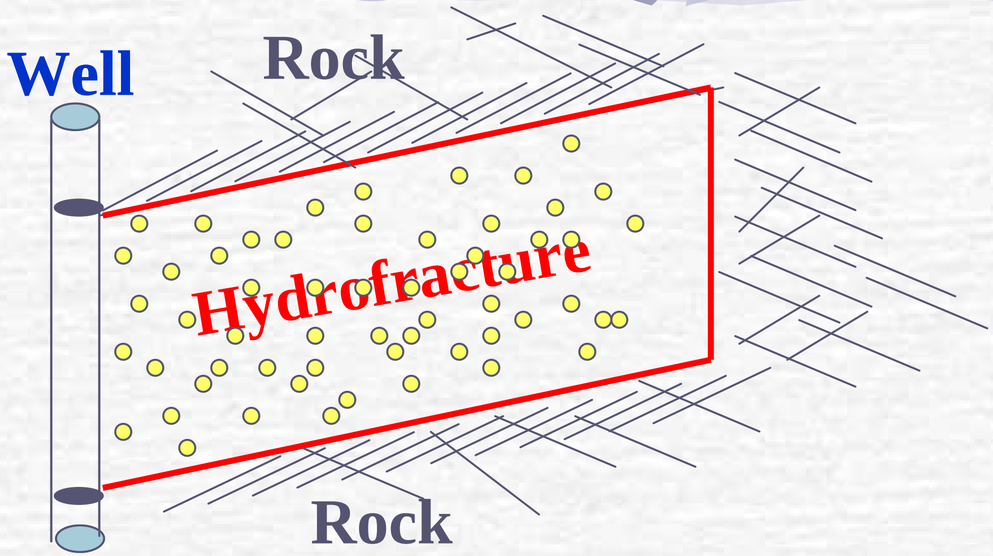
Essence of Hydraulic Fracturing



ESSENCE

Drastic increase of the surface,
to which oil (gas, heat) flows

Growing Importance of Hydraulic Fracturing

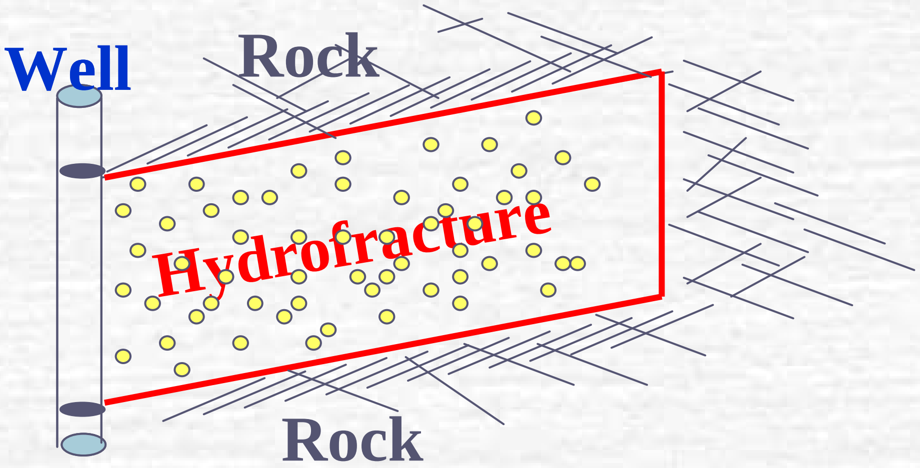


The importance of hydraulic fracturing
has dramatically grown last years
because huge resources of gas
are found in low permeable shales

The key element of technology, used in shales, is hydraulic fracturing

Hydraulic Fracturing

Other Applications

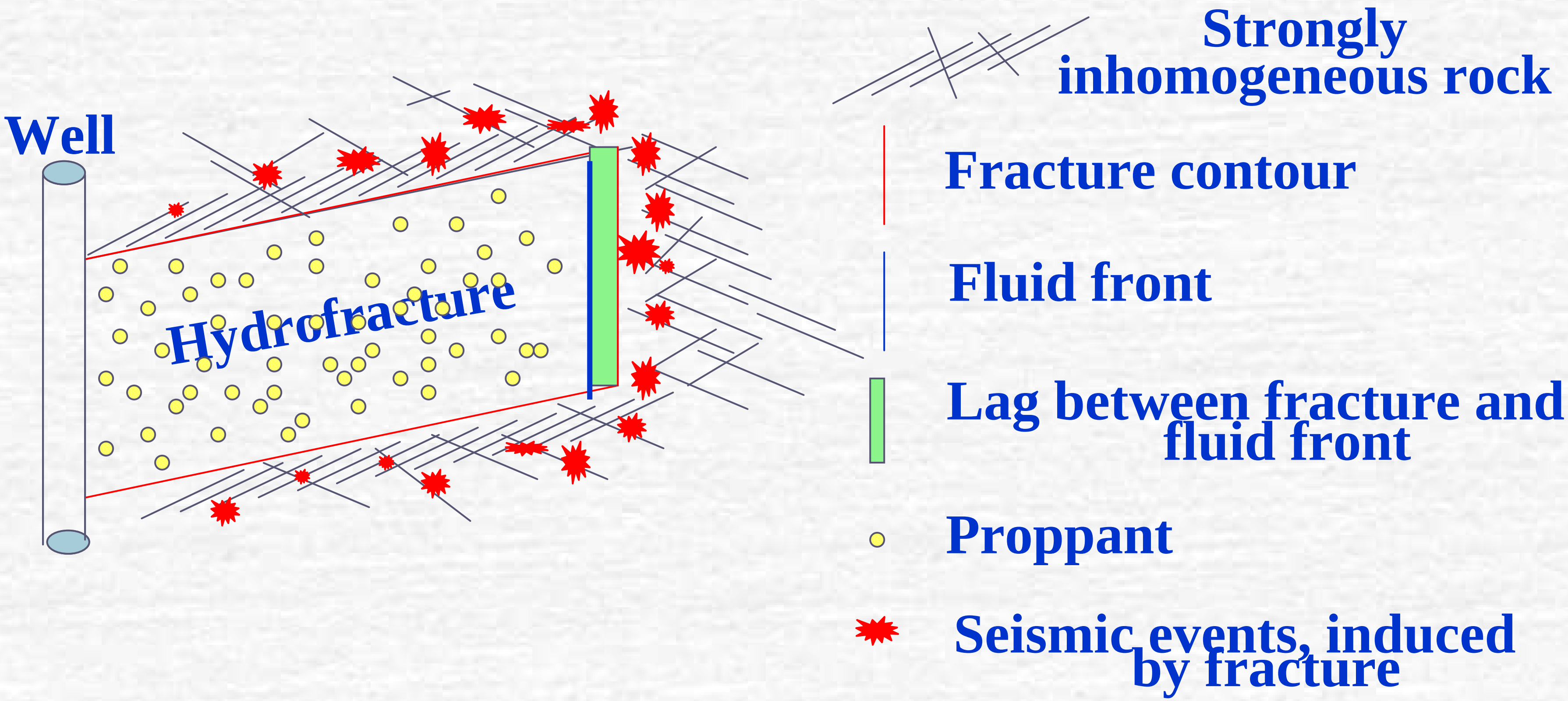


Hydraulic fractures are also used to

- Increase heat production from geothermal reservoirs
- Measure in-situ stresses
- Control caving of roof in coal and ore excavations
- Enhance CO_2 sequestration
- Isolate toxic substances in rock

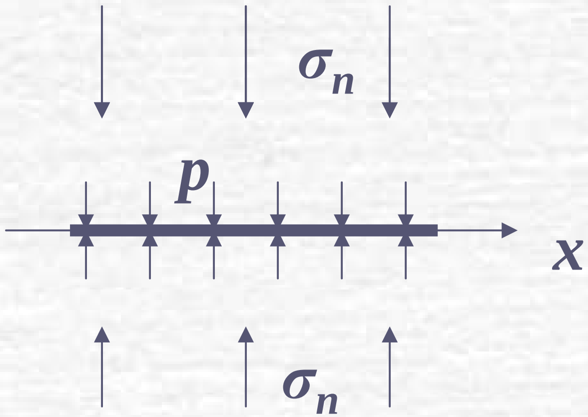
***In natural conditions*, pressurized melted substance fractures earth crust leading to formation of veins of mineral deposits**

Scheme Explaining Problems of Modeling Hydraulic Fractures



To efficiently employ hydraulic fracturing,
we need to properly model it
accounting for the most essential features listed

First Theoretical Models

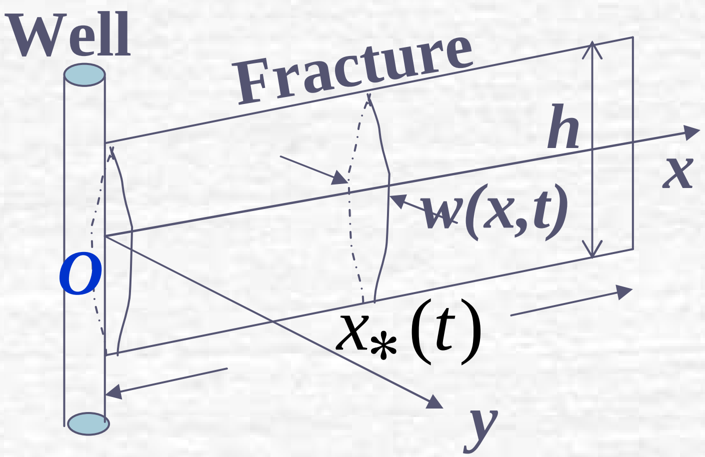


KGD model:

plane-strain state in *horizontal* cross-sections

Khristianovich & Zheltov 1955

Geertsma & de Klerk 1969



PKN model:

plane-strain state in *vertical* cross sections

Perkins & Kern 1961

Nordgren 1972

Further Theoretical Work

Studying of asymptotics and self-similar solutions

Numerous papers on theoretical studying of hydraulic fracturing are focused on

(i) asymptotics at crack tip;

(ii) self-similar and asymptotic solutions to study regimes of flow

Spence & Sharp 1985: self-similar plane problem and asymptotics for newtonian liquid;

Desrouches, Detournay et al 1994: asymptotics for power-law liquid;

Adachi & Detournay 2002: self-similar plane problem for power-law liquid;

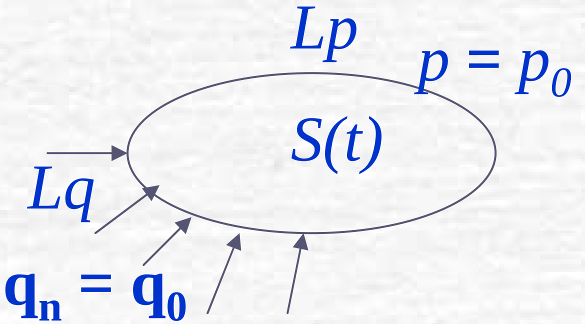
Savitski & Detournay 2002: self-similar axisymmetric problem for Newtonian liquid;

Michell, Kuske & Pierce 2007: asymptotics and regimes

Hu & Garagash 2010: plane problem; accounting for leak-off

Conventional Formulation

Equations for fluid



Continuity equation

$$div\boldsymbol{q} + \partial w/\partial t - q_e = 0 \qquad \textbf{(1)}$$

Poiseuille equation

$$\boldsymbol{q} = -D(w,p)\mathrm{grad}p \qquad \textbf{(2)}$$

Reynolds equation (using (2) in (1))

$$div[D(w,p)\mathrm{grad}p] - \partial w/\partial t + q_e = 0 \qquad \textbf{(3)}$$

Initial condition (zero opening) $\quad w(x,0) = 0 \qquad$ **(4)**

BC from physical considerations (at the fluid contour)

$$q_n(\boldsymbol{x}) = q_0(\boldsymbol{x}) \quad \boldsymbol{x} \in L_q \qquad p(\boldsymbol{x}) = p_0(\boldsymbol{x}) \quad \boldsymbol{x} \in L_p \qquad \textbf{(5)}$$

EMPHASIZE THAT

the conventional formulation employs the

flux q

rather than the

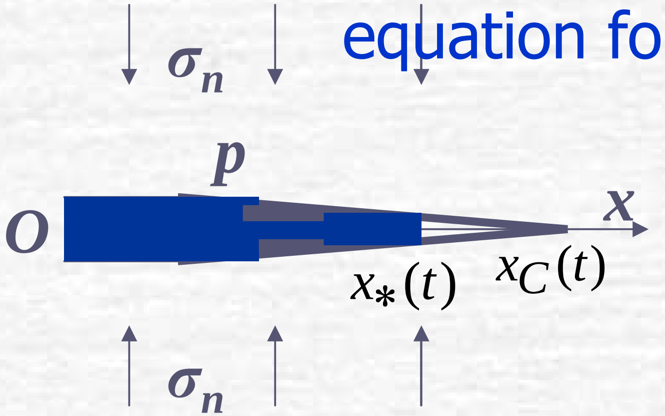
fluid particle velocity,

despite the particle velocity is the primary quantity used

when deriving the Poiseuille equation

Equations for Solid

The opening *w* being unknown, we need a solid mechanics equation for embedding solid (rock)



Solid mechanics equation

(commonly BIE of linear elasticity)

$$A(w, p) = 0$$

Boundary condition ***(at crack contour)*** $\quad w(\mathbf{x_c}) = 0$

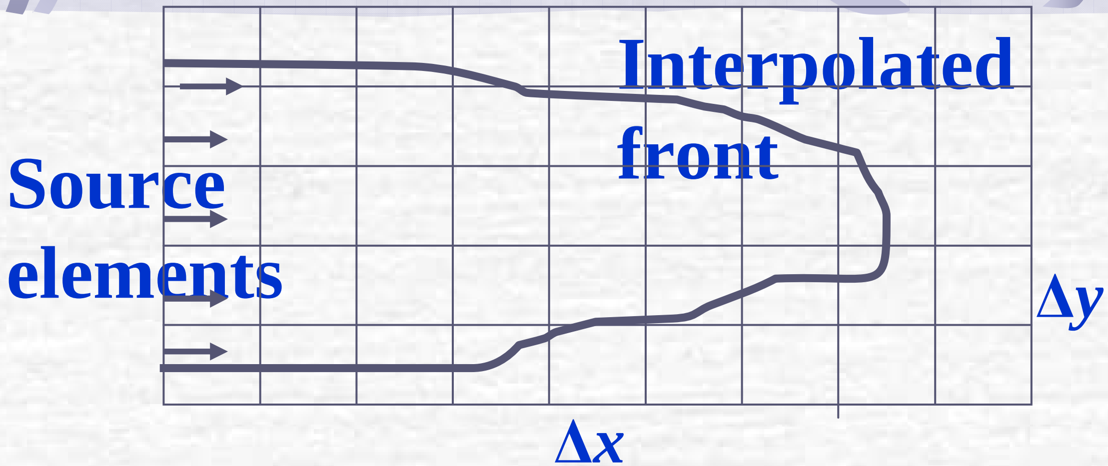
Fracture mechanics strength equations

(commonly in terms of SIFs)

$$K_I = K_{I\mathbf{c}}$$
$$K_{II} = 0$$

Strength limitation permits crack propagation.
In general, it also defines the *lag*
between the fluid front and the crack tip

Simulators of Hydraulic Fractures



Planar fracture geometry based on rectangular boundary elements

Simulators

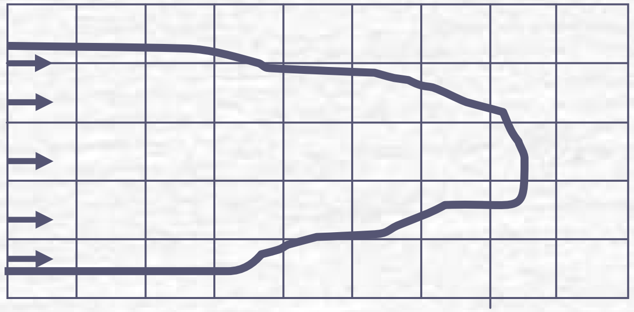
USA: Schlumberger (Siebrits et al; Cipola et al.), Pinnacle (Warpinski)
USA: (Cleary et al)
Japan: (Jamamoto et al.)

}? Black boxes

Inexplicitly, numerics built in Schlumberger codes is sketched in:
Adachi, Siebrits et al, *Int. J. Rock Mech Min. Sci.*, 2007, 44, 739-757

The authors emphasized the need
"to dramatically speed up … simulators"

Means to Meet Challenge



"To dramatically speed up simulators"

it looks reasonable to employ methods of the

THEORY OF PROPAGATING INTERFACES

J. A. Sethian, *Level Set Methods and Fast Marching Methods*, Cambridge, Cambridge Univ. Press, 2nd ed., 1999

The basic concept of the theory is

SPEED FUNCTION

BUT! For more than 40 years, it has not been employed for hydraulic fracture simulation

"WHY NOT?"

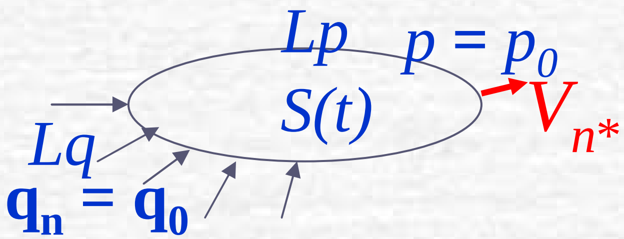
Hydraulic Fracturing

PART II

ANSWER TO THE QUESTION
“WHY NOT?”
AND
NEW RESULTS

Why Not?

Fluid Flux vs. Particle Velocity



The SPEED FUNCTION is the VELOCITY V_{n*} ***of the fracture front.***

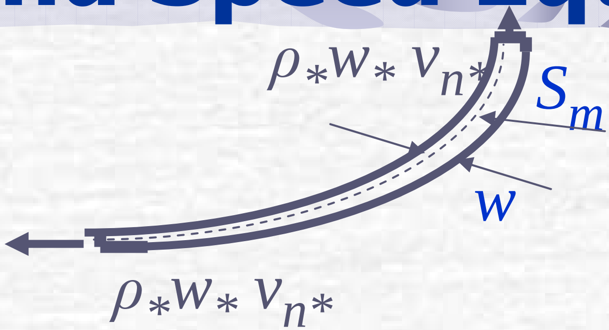
Hence we need a velocity.

BUT !

As mentioned, the conventional formulation employs the fluid flux rather than the fluid particle velocity. Meanwhile, the particle velocity is the primary quantity used when deriving the continuity and Poiseuille equations

This indicates that it is reasonable to revisit basics

Reynolds Transport Theorem and Speed Equation



Hydrofracture is a *narrow channel*

Reynolds transport theorem for flow in a *narrow channel*:

$$\frac{dM_e}{dt} = \int_{S_m(t)} \frac{\partial(\rho w)}{\partial t} dV + \int_{L(t)} \rho w v_n dS \quad \Rightarrow \quad \dot{m}_e = \frac{\partial(\rho w)}{\partial t} + \frac{\partial(\rho w v_k)}{\partial x_k}$$

continuity eqn

Note that the flux appears only after setting *by definition* $\mathbf{q} = \rho w \mathbf{v}$

For the *entire* volume, occupied by a fluid, the integral form reads:

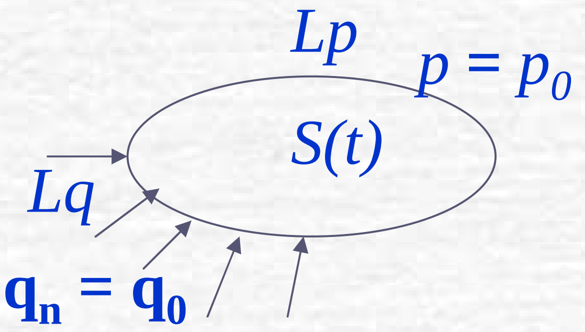
$$\frac{dM_e}{dt} = \int_{S_t} \frac{\partial(\rho w)}{\partial t} dV + \int_{L_*(t)} \rho_* w_* v_{n*} dS$$

By derivation, the particle velocity on the front equals the speed of propagation. Therefore:

$$V_{n*} = \frac{dx_{n*}}{dt} = v_{n*} = \frac{q_{n*}}{\rho_* w_*}$$ **is the needed Speed Equation (SE)**

$$F = v_{n*} = \frac{q_{n*}}{\rho_* w_*}$$ **is the needed Speed Function (SF)**

Particular Feature of Conventional Formulation



Continuity equation (local form)

$$div\boldsymbol{q} + \partial w/\partial t - q_e = 0 \qquad \textbf{(1)}$$

Poiseuille equation

$$\boldsymbol{q} = -D(w, p)\mathrm{grad}p \qquad \textbf{(2)}$$

Reynolds equation (using (2) in (1))

$$div[D(w, p)\mathrm{grad}p] - \partial w/\partial t + q_e = 0 \qquad \textbf{(3)}$$

Initial condition (zero opening) $w(x,0) = 0$ **(4)**

BC from physical considerations (at the liquid contour)

$$q_n(\boldsymbol{x}) = q_0(\boldsymbol{x}) \quad \boldsymbol{x} \in L_q \qquad p(\boldsymbol{x}) = p_0(\boldsymbol{x}) \quad \boldsymbol{x} \in L_p \qquad \textbf{(5)}$$

But !We have additional SPEED EQUATION (at the fluid contour)

BC=SE ! $$v_{n*} = \frac{q_*}{w_*} = -\frac{1}{w_*} D(w, p)\frac{\partial p}{dn_*} \qquad x \in L_q + L_p \qquad \textbf{(6)}$$

Thus for the *elliptic* (in spatial coordinates) *operator*

we have *two* rather than one *boundary conditions*

involving *a function and its normal derivative*

This indicates that there might be difficulties

Specifically, *for a fixed front, the problem appears ill-posed*

Hadamard Definition and Tychonoff Regularization

By Hadamard, a problem is *well-posed* when

- **A solution exists**
- **The solution is unique**
- **The solution depends continuously on the data, in a reasonable metric**

Jacques Hadamard (1902), *Sur les problemes aux derivees partielles et leur signification physique*, Princeton Univ. Bul. 49-52

Otherwise, a problem is *ill-posed*

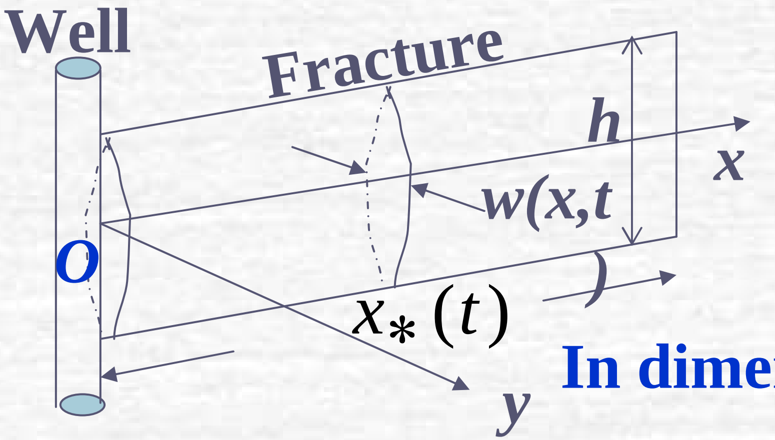
Hadamard considered that ill-posed problems had no physical sense

A.N. Tychonoff (1943) clearly recognized significance of ill-posed problems for applications. He was the first to suggest a means to solve them numerically by using *regularization:*

A.N. Tychonoff (1963) *Solution of incorrectly formulated problems and the regularization method*, Soviet Mathematics 4, 1035-1038. [Transl. from Russian: А. Н. Тихонов, ДАН СССР, 1963, 151, 501-504]

We need a proper method of regularization for the problem of hydraulic fracturing

Clear Evidence that BVP is Ill-Posed: *Nordgren Problem*



Elasticity equation for plane-strain in vertical cross-sections $p = k_r w$

Reynolds equation (Newtonian liquid)

$$k_l \frac{\partial}{\partial x}\left(w^3 \frac{\partial p}{\partial x} \right) - \frac{\partial w}{\partial t} = 0$$

In dimensionless variables, the problem becomes

$$\frac{\partial^2 w^4}{\partial x^2} - \frac{\partial w}{\partial t} = 0$$ Nordgren's PDE

Initial condition: $w(x,t_0)=w_0(x)$

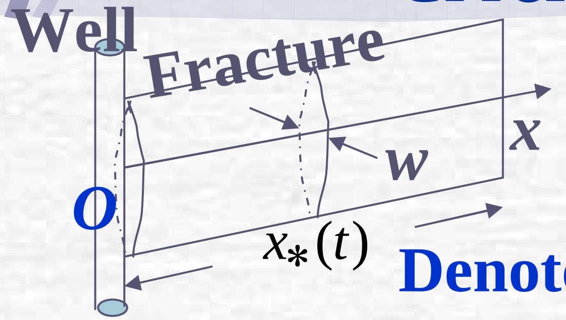
Boundary conditions: $-\left.\frac{\partial w^4}{\partial x}\right|_{x=0} = \boldsymbol{q}_0$ BC at inlet $x = 0$

$w(\boldsymbol{x}_*, t) = 0$ BC at liquid front $x = x_*$

+ Speed Equation: $V_* = \frac{dx_*}{dt} = -\frac{4}{3}\frac{\partial w^3}{\partial x}\Big|_{x=x_*(t)}$

There are *three* rather than *two* BC for the PDE of *second* order in spatial variable *x*. For any fixed x_*, the *problem is ill-posed*

Even *More* Clear Evidence that BVP is Ill-Posed



The Nordgren problem is self-similar.
Introduce self-similar variables

$$x = \xi t^{4/5}, \quad w(x) = t^{1/5}\psi(xt^{-4/5}) \qquad x_* = \xi_* t^{4/5}$$

Denote $y(\xi) = \psi^3(\xi)$ **The problem is reduced to ODE**

$$\frac{d^2 y}{d\xi^2} + a(y, dy/d\xi, \xi)\frac{dy}{\partial\xi} - \frac{3}{20} = 0 \qquad \textbf{ODE} \qquad (1)$$

where $a(y,dy/d\xi,\xi)=(dy/d\xi+0.6\xi)/(3y)$ **is finite at fluid front** $\xi=\xi_*$

Boundary conditions for the ODE of *second* order:

$$\left.\frac{dy}{\partial\xi}\right|_{\xi=0} = -0.75\frac{q_0}{\sqrt[3]{y(0)}} \qquad \textbf{BC at inlet } \xi = 0 \quad (2)$$

$$y(\xi_*) = 0 \qquad \textit{BC at fluid front } \xi = \xi_* \quad (3)$$

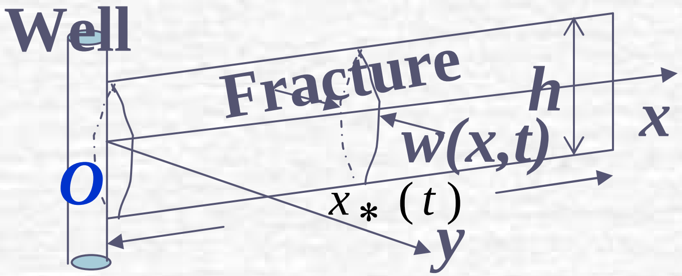
+ *SPEED EQUATION,* which is met identically by a solution of ODE satisfying BC (3):

$$\left.\frac{dy}{\partial\xi}\right|_{\xi=\xi_*} = -0.6\xi_* \qquad \textit{SE at fluid front } \xi = \xi_* \quad (4)$$

Thus, there are *two, rather than one,* BC at the fluid front. By Picard, theorem, the *Cauchy conditions* (3), (4) uniquely define $y(\xi)$, $dy/d\xi$ **and consequently influx at the inlet. Hence, *a solution of BVP (1)-(3) does not exist for an arbitrary influx.***

By Hadamard definition, *the BV problem (1)-(3) is ill-posed*

Solution of Nordgren Problem
without regularization



We solved both the starting and self-similar BV Nordgren problem by finite differences
without regularization

By no means could we have more than two correct digits

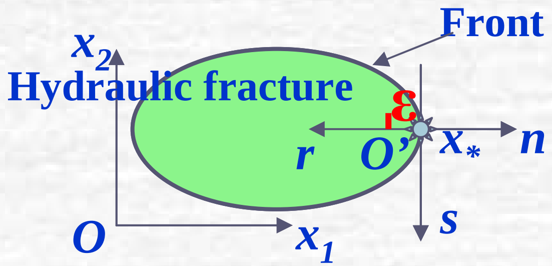
Furthermore,

- The results always deteriorated ***near the front***
- Using fine meshes (with the step less than $10^{-5}x_*$) led to ***complete deterioration*** of the solution ***in the entire region***

This clearly shows that a proper regularization method is needed to have accurate and reliable numerical results

Regularization Method for Hydraulic Fracturing

We suggest the regularization method employing the very cause of the difficulty



We have:

PDF $$\frac{\partial w}{\partial t} - \mathrm{div}\big(D(w,p)\,\mathrm{grad}p\big) - q_e = 0 \qquad (1)$$

with **two** *BC at a point* x_* *of the liquid front*

$$p(\mathbf{x}_*) = p_0(\mathbf{x}_*) \qquad (2)$$ *Prescribed for a problem*

$$-\frac{1}{w_*(\boldsymbol{x}_*)} D(w,p) \frac{\partial p}{\partial n}\bigg|_{x=x_*} = v_{n*} \qquad (3)$$ *Speed Equation*

Integration of (3) and accounting for (2) yield

$$\int_{p_0}^{p} \frac{1}{w} D(w,p)\,dp \approx v_* r \qquad (4)$$

By using (4) we impose the BC at a small distance ε behind the front:

$$\int_{p_0}^{p_\varepsilon} \frac{1}{w} D(w,p)\,dp = v_* \varepsilon \qquad (5)$$

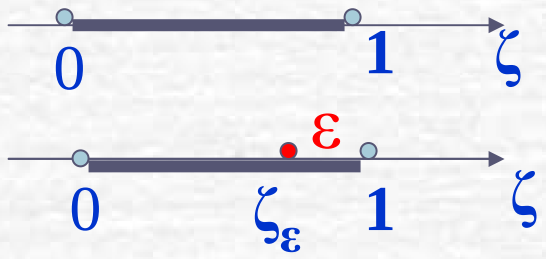
The regularization method consists in using the BC (5) at a small distance ε behind the front rather than the BC (2) and (3) on the front

We call this approach *ε - regularization*

It appears really efficient for solving HF problems

Solution of Nordgren Problem

with *ε – regularization*



We have obtained that near the front:

$$Y(\varsigma,t) \approx 0.75 x_*(t) v_*(t)(1-\varsigma)$$

Hence, we may impose the BC at the relative distance *ε* behind the front

$$Y(\varsigma_\varepsilon,t) = 0.75 x_*(t) v_*(t)\varepsilon$$

We solved both the starting and self-similar BV Nordgren problem by finite differences *with ε -regularization*

Conclusions obtained:

- **The *results are accurate* in a wide range of *ε* ($10^{-2} > \varepsilon > 10^{-5}$), size ($10^{-2} > \Delta\zeta > 10^{-5}$) and number (up to 100 000) of time steps**

  For ODE of self-similar formulation, there are **six correct digits**, at least;

  For PDE, the error is less than 0.03% even for **100 000 steps**

- **There are *no signs of instability* in specially designed experiments**

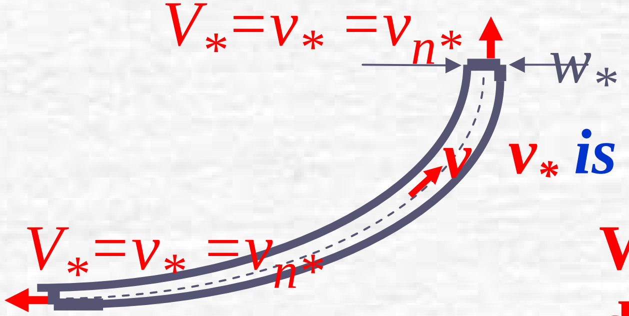
- ***Small time expense* on a conventional laptop**

  Even for 100 000 steps, the time expense does not exceed 15 s

This shows that ε - regularization is efficient

There are also other important implications of the SE concerning with a proper choice of variables

Importance of Particle Velocity



$$V_{n*} = v_{n*} = \frac{q_{n*}}{w_*}$$ **Speed *Equation* (SE)**

v_* is the value of particle velocity v at the fluid front

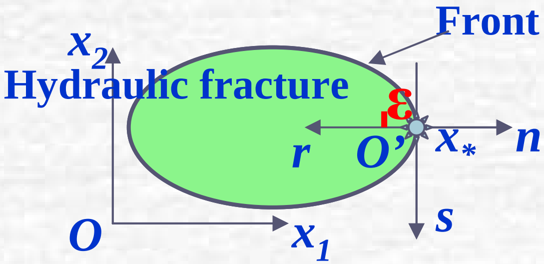
We see that, although the particle velocity does not enter the conventional formulation, *it is of physical and mathematical significance:*

- It defines the speed of the front propagation,
- It is the primary quantity defining the flux $q = \rho w v$,
- It also defines the movement of proppant, used to prevent the fracture closure,
- It is *non-zero* and *finite* function in the entire flow region, what makes it a proper choice as a convenient mathematical quantity

Thus, it looks reasonable to reformulate the HF problem by including the particle velocity into equations

Proper Choice of Variables

We have already noticed that the SE and regularization method suggest using:



(i) Particle velocity $v = \frac{1}{w} D(w, p)\mathrm{grad}p$ ***instead of*** **pressure *p***

(ii) Local coordinates *r, s* moving with the front *instead of* the fixed system x_1x_2

Besides, the SE, taken together with an elasticity eqn, implies that commonly the opening has power asymptotics $w = C(t)r^\alpha$, with $0 < \alpha < 1$.

Hence its *derivative is singular* near the front: $dw/dr \to \infty$, when $r \to 0$.

Thus, it is reasonable to use:

(iii) Modified opening $y = w^{1/\alpha}$ instead of opening *w*

In new variables, we obtain the modified lubrication equation:

$$\frac{\partial y}{\partial t} = \frac{y}{\alpha}\frac{\partial v_n}{\partial r} + (v_n - V*)\frac{\partial y}{\partial r} - \frac{y^{1-\alpha}}{\alpha} q_l$$

Emphasize that the new variables *v* and *y* have 'good' properties:

- Particle velocity is ***non-zero finite smooth*** function up to the front
- Modified opening is ***linear*** near the front

Thus we have obtained the *modified formulation* of the HF problem

Modified Formulation

Summarizing, we come to the *modified formulation* of HF problem. In contrast with the conventional formulation, it uses:

- The ***particle velocity***, as a variable smooth near the front,
  *instead of the **pressure***;
- The ***modified opening***, which is linear near the front,
  *instead of the **opening** itself*;
- The ***SE*** at each point of the front,
  *instead of the single equation of **global mass balance***;
- ***ε-regularization*** to exclude solution deterioration caused by the fact that the problem is ill-posed for a fixed position of the front;
- ***Moving spatial coordinates***;
- ***Reformulation*** of the common system ***of equations and BC*** in terms of the suggested variables complimented, when appropriate, with ε-regularization.

Computational Advantages of Modified Formulation

Computational advantages **have been explained and illustrated by revisiting the classical Nordgren problem.**

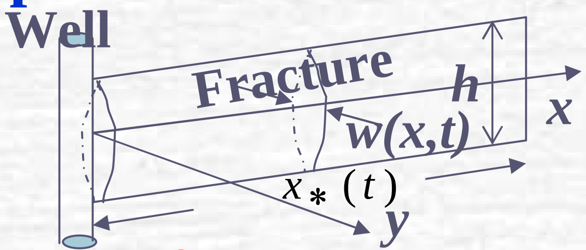
The main advantages are:

- Possibility to use well-established methods of the theory of propagating interfaces;
- Avoiding deterioration of numerical solution;
- Avoiding singularities at the fluid front.

Analytical Advantages of Modified Formulation

Analytical advantages **are evident when revisiting the classical problems**

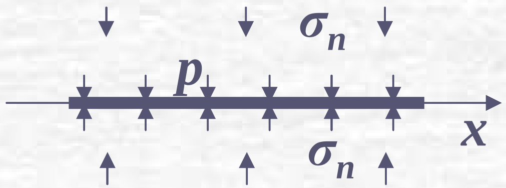
***PKN model*: plane-strain state in *vertical* cross section**



The conventional formulation requires involved calculations

See: Nordgren, Soc. Pet. Eng., 1972, August, 306-314

***KGD model*: plane-strain state in *horizontal* cross-section**



Again, the conventional formulation requires involved calculations

See: Spence & Sharp, Proc. Roy. Soc. London, A, 1985, 400, 289-313;

Adachi & Detournay, Int. J. Numer. Anal. Meth. Geomech., 2002, 26, 579-604

For both problems, the modified formulation provides simple analytical solutions

*See: Linkov, IJES, 2012, **52**, 77-88*

Further Work

Further work may employ new options provided by the modified formulation. They include:

- Development of new efficient algorithms for simulation of HF;
- Improving commercial codes serving for modeling HF;
- Obtaining analytical solutions accounting for leak-off and non-Newtonian behavior of fracturing fluids;
- Proper accounting for the lag between the fluid front and the crack contour;
- Proper modeling of proppant movement

The work is in progress.
Hopefully, the joined efforts
of English, Polish and Russian colleagues
will provide useful results in the areas listed

Thank you!