Speed Equation and its Application for Solving Ill-Posed Problems of Hydraulic Fracturing

A. M. Linkov

Presented by Academician N. F. Morozov March 17, 2011

Received March , 2011

Mathematical modeling of a fluid driven fracture, first discussed in [1], is of prime significance for hydraulic fracturing. Models developed to date employ the integral form of global mass balance (e.g. [2 - 5]). We demonstrate that using the local form, called the speed equation, shows specific features of the problem: it is ill-posed when considered as a boundary value (BV) problem. The equation also provides a means to regularize the problem and solve it efficiently.

Initially, we show that the speed equation is fundamental in the sense that it does not depend on a particular shear law of a liquid. When applied to a narrow channel between closely located boundaries, the mass conservation equation for an incompressible liquid is

$$\frac{dVe}{dt} = \int_{S_l} \frac{\partial w}{\partial t} dS + \int_{L_l(t)} w_*(\boldsymbol{x}_*) v_{n*}(\boldsymbol{x}_*) dL , \qquad (1)$$

where S_l is the middle surface, w is the height (opening) of the channel, $L_l(t)$ is the contour of the liquid front at the time t, x_* is a point on the front, v_{n*} is the normal to L_l component of the fluid particle velocity averaged across the height. Note that in (1), the average particle velocity v_{n*} also represents the speed of the front propagation. As $q_{n*}(x_*) = w_*(x_*)v_{n*}(x_*)$ is the flux through the front cross-section, we obtain the fundamental equation which gives the front velocity as a function of the flux and opening:

$$v_{n*}(\boldsymbol{x}_*) = \frac{q_{n*}(\boldsymbol{x}_*)}{w_*(\boldsymbol{x}_*)}. \tag{2}$$

Use the Reynolds equation for flow of viscous incompressible liquid in a narrow channel:

$$\frac{\partial w}{\partial t} - \frac{\partial}{\partial x_i}\left(D(w,p)\frac{\partial p}{\partial x_i} \right) - q_e = 0. \tag{3}$$

where D is a prescribed function or operator; p is the pressure, averaged through the cross-section; v_i (i = 1,2) are components of the average velocity of liquid particles in a channel cross section; the Cartesian coordinates x_1, x_2 are located in the fracture plane. Non opening fracture along a crack trajectory is assumed as an initial condition when studying hydraulic fracture. The boundary condition on the liquid front is the condition of the prescribed flux q_0 at a part L_q and of the prescribed pressure p_0 at the remaining part L_p of the contour L_l:

$$q_n(\boldsymbol{x}) = q_0(\boldsymbol{x}) \quad \boldsymbol{x} \in L_q; \qquad p(\boldsymbol{x}) = p_0(\boldsymbol{x}) \quad \boldsymbol{x} \in L_p. \tag{4}$$

The opening in (3) being unknown, we need elasticity equation connecting the opening w and pressure p. Additionally, the criterion of linear fracture mechanics is imposed: $K_I = K_{IC}$, where K_I is the stress intensity factor, K_{IC} is its critical value.

In view of (2), prescribing the boundary conditions (4) means that there are two conditions at the points of a liquid front. This leads to difficulties common to over-determined problems [7-9] when solving the problem numerically, because the boundary is fixed on iteration. To find a means to overcome the difficulties, we study the Nordgren problem [2]. The Nordgren model considers straight fracture along the x-axis (Fig.) with the assumption that the pressure p is proportional to the opening w. Neglecting liquid leak-off and normalizing the variables, the equation (3) reads [2]:

$$\frac{\partial^2 w^4}{\partial x^2} - \frac{\partial w}{\partial t} = 0. \tag{5}$$

The boundary conditions include the prescribed normalized flux q_0 at the inlet $x = 0$:

$$\frac{\partial w^4}{\partial x} = -q_0 \tag{6}$$

and zero opening (and flux) at the liquid front $x = x_*$, which coincides with the crack tip:

$$w(x_*) = 0. \tag{7}$$

The opening is assumed positive for $0 \le x < x_*$. We shall also use the speed equation (2) which becomes:

$$v_* = -\frac{4}{3}\frac{\partial w^3}{\partial x}\Big|_{x=x_*} . \tag{8}$$

The problem being self-similar, the solution is represented as $w = t^{1/5}\psi(\xi)$, where $\xi = xt^{-4/5}$, so that $x = \xi t^{4/5}$, $x_* = \xi_* t^{4/5}$, $v_* = dx_* / dt \ = 0.8\xi_* t^{-1/5}$, ξ_* is the automodel coordinate of the liquid front depending only on the prescribed flux q_0. Then the equation (5) becomes the ordinary differential equation:

$$\frac{d^2 y}{d\xi^2} + a(y, dy/d\xi, \xi)\frac{dy}{\partial\xi} - \frac{3}{20} = 0, \tag{9}$$

where $y(\xi) = \psi^3(\xi)$, $a(y, dy/d\xi, \xi) = (dy/d\xi + 0.6\xi)/(3y)$.

The boundary conditions (6) and (7) read:

$$\frac{dy}{\partial\xi}\Big|_{\xi=0} = -0.75\frac{q_0}{\sqrt[3]{y(0)}}, \tag{10}$$

$$y(\xi_*) = 0, \tag{11}$$

and the speed equation (8) becomes:

$$\frac{dy}{\partial\xi}\Big|_{\xi=\xi_*} = -0.6\xi_* . \tag{12}$$

It is easily shown that $C_* = (q_0)^{0.6} / \xi_*$ and $C_0 = y(0)/\xi_*^2$ are constants independent of the flux q_0. Since $\xi_* = (q_0)^{0.6} / C_*$, we may prescribe q_0 or ξ_*, as convenient. A particular value of q_0 or ξ_* may also be conveniently taken.

We can now fix ξ_*. Then according to (11), (12), at a fixed point ξ_*, we have prescribed both the function y and its derivative $dy/d\xi$. Thus, for the equation of the second order (9) we have a Cauchy problem. Its solution defines $y(0)$ and $dy/d\xi|_{\xi=0}$ and consequently the flux q_0 at $\xi=0$. A small error when prescribing q_0 in (10) excludes the existence of the solution of the BV problem (9)-(11). By definition [7], the BV problem (9)-(11) is ill-posed and needs regularization [8, 9].

Conversely, the Cauchy problem (9), (11), (12) is well-posed and leads to a bench-mark solution. We obtained the solution by applying the fourth order Runge-Kutta scheme to the system of two differential equations in unknowns $y_1(\xi)=y(\xi)$, $y_2(\xi)=dyl\xi$, equivalent to (9). The constants C_* and C_0 evaluated with seven significant digits are: $C_*=0.7570913$, $C_0=$ 0.5820636. For the value $q_0=2/\pi$, used by Nordgren [2], we have $\xi_*=1.0073486$, $\psi(0)=0.8390285$ against the values $\xi_*=1.01$, $\psi(0)=0.83$ given by this author with the accuracy of about one percent. Bench-mark values of the function $y(\xi)$ and its derivative served us to evaluate the accuracy of further calculations obtained by using various approaches.

We could see that when solving the BV problem (9)-(11) it is impossible to obtain more than two correct digits. What is notable, this level of accuracy was obtained even when using a rough mesh with only one-hundred nodes. This implies that using a rough mesh may serve to regularize the problem when high accuracy is not needed. For fine meshes, we could see strong deterioration of the results near the liquid front $\xi = \xi_*$.

Likewise, our attempts to accurately solve the problem (5)-(7) also failed when using time steps with finite difference approximations for $\partial^2 w / \partial x^2$ and $\partial w / \partial x$ at a step. By no means could we have three correct digits, and the results always strongly deteriorated near the liquid front $x = x_*(t)$. Again, fine meshes did not improve the accuracy as compared with a rough mesh having the step $\Delta\varsigma = \Delta x / x_* = 0.01$.

The experiments confirm that the ill-posed problem under consideration cannot be solved accurately without regularization. A regularization method is suggested by the conditions (11), (12). Indeed, they yield the approximate equation $y \approx 0.6\xi_*(\xi_* - \xi)$ near the front. Hence, instead of prescribing a boundary condition at the front $\xi = \xi_*$, we impose it at a point $\xi_\varepsilon = \xi_*(1-\varepsilon)$ at a small relative distance ε from the front:

$$y(\xi_\varepsilon) = 0.6\xi_*^2\varepsilon. \tag{13}$$

The BV problem (9), (10), (13) is well-posed; it may be solved by finite differences. It appears that with ε = 10^{-3}, 10^{-4}, the results for the steps $\Delta\varsigma = \Delta\xi/\xi_*$ = 10^{-3}, 10^{-4}, 10^{-5}, 10^{-6} coincided with those provided by the bench-mark solution. The results are stable if ε and $\Delta\varsigma$ are not simultaneously too small ($\varepsilon, \Delta\varsigma > 10^{-5}$). However, as expected, the results deteriorate when both ε and $\Delta\varsigma$ are too small; they become absolutely wrong when $\varepsilon = \Delta\varsigma = 10^{-6}$. We could also see that as ε increases, the accuracy decreases and it actually does not depend on the step if the latter is small enough. In particular, for the step $\Delta\varsigma$ = 0.1, the accuracy is one percent for ε = 0.01, and the results stay at the same accuracy level even for ε = 10^{-9}.

The suggested regularization consists in using the speed equation together with a prescribed boundary condition to formulate the boundary condition at a small relative distance ε behind the front rather than on the front itself. We call such an approach ε-regularization. It is applicable in general 1D and 2D cases when a self-similar formulation is not available or is not used. To illustrate, we employed the ε - regularization for the starting equation (5) under the boundary conditions (6), (7). In terms of the variable $Y = w^3$, the prescribed condition (7) and the speed equation (8) yield $Y(x,t) \approx 0.75 x_*(t) v_*(t)[1 - x / x_*(t)]$ at points close to the front. Hence, the boundary condition at a point $x_\varepsilon = x_*(1-\varepsilon)$ with the relative distance ε from the front is:

$$Y(x_\varepsilon, t) = 0.75 x_*(t) v_*(t) \varepsilon \,. \tag{14}$$

Thus, the regularized problem consists in solving (5) under zero-opening initial condition and the boundary conditions (6) and (14). Numerical experiments have shown that the ε-regularization removes the difficulties and provides accurate results.

The conclusions of the paper are as follows: (i) the derived speed equation may serve for tracing hydraulic fracture by methods of the theory of propagating surfaces; (ii) when simulating hydraulic fracture numerically, it is useful to employ the ε - regularization consisting in prescribing a boundary condition at a small relative distance ε behind the front; (iii) the method provides an efficient means for solving problems of hydraulic fracture.

ACKNOWLEDGEMENTS

The author appreciates the support of the EU Marie Curie IAPP program (Grant # 251475).

REFERENCES

1. Khristianovich S. A., Zheltov V. P., Proc. 4-th World Petroleum Congress, Rome, 1955, 579-586 .
2. Nordgren R. P., Soc. Petroleum Eng. J., 1972, **12**(8), 306-314.
3. Spence D. A., Sharp P. W., Proc. Roy Soc. London, Ser. A, 1985, **400**, 289-313.
4. Adachi J., Siebrits E., et al., Int. J. Rock Mech. Mining Sci., 2007, **44**, 739-757.
5. Hu J., Garagash D. I., J. Engineering Mech., ASCE, 2010, **136**(9), 1152-1166.
6. Sethian J. A., *Level Set Methods and Fast Marching Methods*, Cambridge, Cambridge University Press, 1999, 370p.
7. Hadamard J., Princeton University Bulletin, 1902, 49-52.
8. Tychonoff A. N., Soviet Mathematics, 1963, **4**, 1035-1038 [Translation from Russian: А. Н. Тихонов, Доклады АН СССР, 1963, **151**, 501-504].
9. Lavrent'ev M. M., Savel'ev L. Ja., *Theory of Operators and Ill-Posed Problems* (Novosibirsk, Institute of Mathematics im. S. L. Sobolev, 1999, 702p), ISBN 5-86134-077-3 [in Russian].

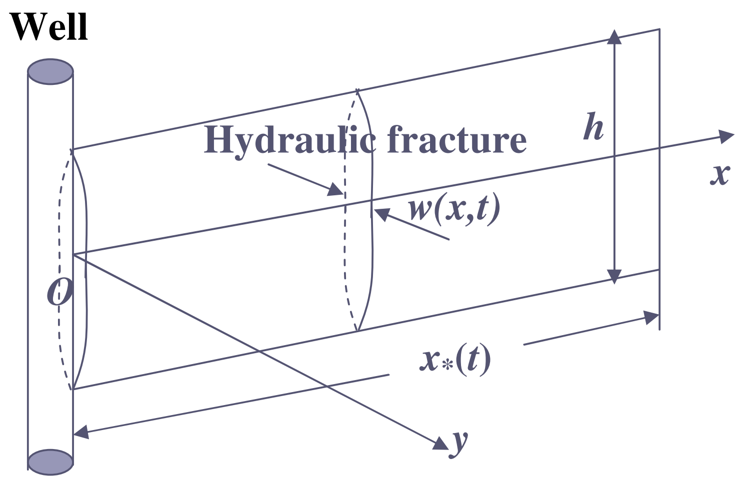

Figure subscription

Fig. Scheme of the problem on hydraulic fracture propagation