Asymptotic solutions for cracks in heterogeneous media
Weight function and dipole matrix approach

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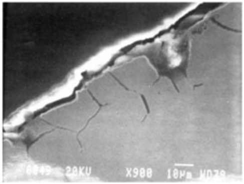
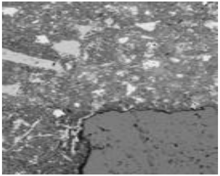
FP7-PEOPLE-2009-IEF Marie Curie - INTERCRACKS

Title:	Unsolved problems in fracture mechanics of heterogeneous materials
Contract Type:	Intra-European Fellowships (IEF)
Researcher:	Dr. A. Piccolroaz
Host Institution:	Aberystwyth University
Coordinator:	Prof. G. Mishuris
Duration:	24 months
Website:	http://fp7.imaps.aber.ac.uk/intercracks.html

Topics of the project





- micromechanical derivation of transmission conditions for different types of interfaces
- derivation of weight functions for cracks along perfect and imperfect interfaces
- interfacial cracks interactions with micro inclusions, inducing crack acceleration and arrest, crack kinking, perturbation of the crack front
- stability analysis of interfacial cracks in dynamic regime
- crack nucleation from, and interaction with, material micro-instabilities
- modelling fracture propagation through lattice structures, keeping into account the interactions between atoms, to be used to disclose the features of failure at the nanoscale

Outline

1. Problem formulation
   - Problem formulation
   - Preliminary results on the unperturbed problem
2. The asymptotic procedure
   - The structure of the asymptotic solution
   - Regular perturbation and dipole fields generated by small defects
   - Singular perturbation for crack advance
   - Analysis of a stable quasi-static propagation
3. Illustrative examples
   - Shielding and amplification effects
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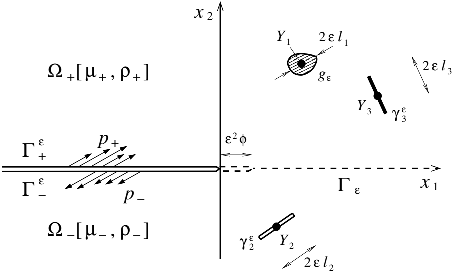
Problem formulation

Bimaterial plane with a dominant crack along the interface and small defects:



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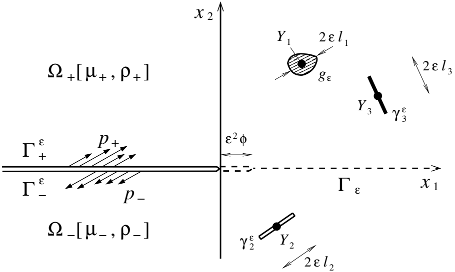

Regular perturbations:

1. Elastic inclusion
2. Microcrack
3. Rigid line inclusion

Singular perturbation:

1. Crack advance

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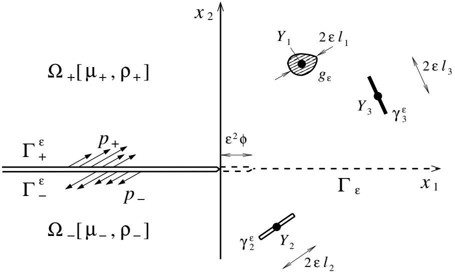

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Small parameter ε:
diameter of defect $2\varepsilon l$, $\varepsilon \ll 1$

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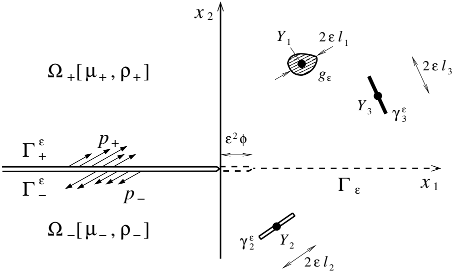

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Assumptions:

1. Perfect interfaces (continuity of displacements and tractions)
2. Linear elastic and isotropic materials
3. The composite is dilute (neglect interactions between small defects)
4. Stable quasi-static propagation (neglect inertia terms)
5. Mode III deformation
6. Loading on the crack surfaces

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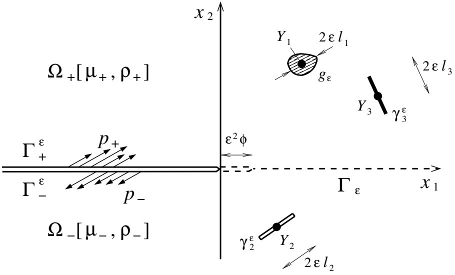

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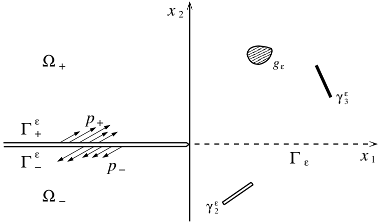
Formulation of the BVP:

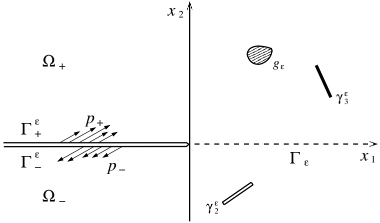


Formulation of the BVP:

- Laplace equation

$$\Delta u_{\pm}(x_1, x_2) = 0 \quad \text{in} \quad \Omega_{\pm}$$

$$\Delta u_i(x_1, x_2) = 0 \quad \text{in} \quad g_{\varepsilon}$$



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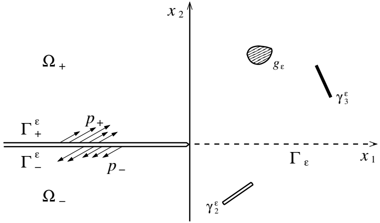

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$$\mu_\pm \frac{\partial u_\pm}{\partial x_2} = p_\pm \quad \text{on } \Gamma_\pm^\varepsilon$$

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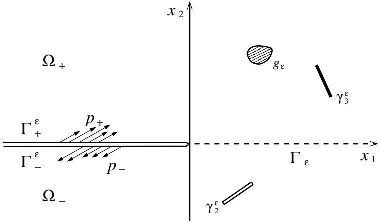

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$$u_+ = u_-, \quad \mu_+ \frac{\partial u_+}{\partial x_2} = \mu_- \frac{\partial u_-}{\partial x_2} \quad \text{on } \Gamma^\varepsilon$$

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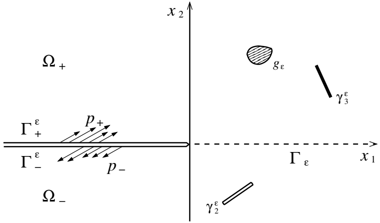

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- Transmission conditions for the elastic inclusion

$$u_+ = u_i, \quad \mu_+ \frac{\partial u_+}{\partial n} = \mu_i \frac{\partial u_i}{\partial n} \quad \text{on } \partial g_\varepsilon$$

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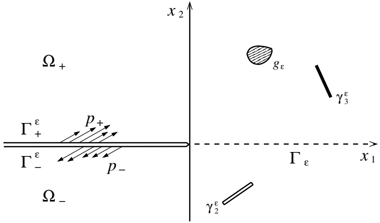

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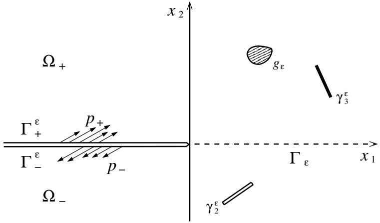

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$$\frac{\partial u_-}{\partial n} = 0 \quad \text{on} \quad \gamma_2^\pm$$

- Dirichlet BC for the rigid line inclusion

$$u_+ = u_* \quad \text{on} \quad \gamma_3^\varepsilon$$

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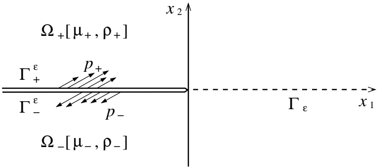
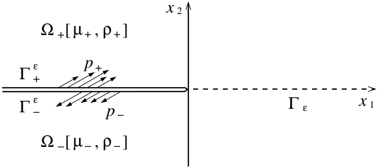


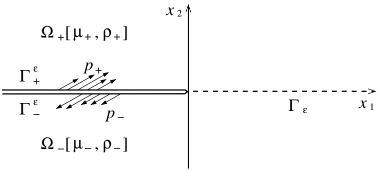
x_2
$\Omega_+[\mu_+, \rho_+]$
Γ_+^ε
p_+
Γ_-^ε
p_-
Γ_ε
x_1
$\Omega_-[\mu_-, \rho_-]$

Preliminary results on the unperturbed problem $\varepsilon = 0$



$\langle p \rangle = (p_+ + p_-)/2$ symmetric loading

$[\![p]\!] = p_+ - p_-$ skew-symmetric loading



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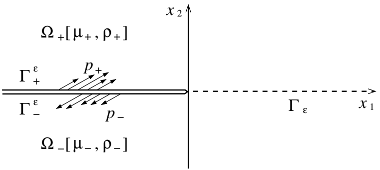
Full-field solution by Mellin transform technique:

$$u_\pm^{(0)}(r,\theta) = \frac{1}{2\pi i} \int_{\omega - i\infty}^{\omega + i\infty} \tilde{u}_\pm^{(0)}(s,\theta) r^{-s} ds, \quad \pm\theta \in (0,\pi), \quad 0 < \omega < 0.5$$

$$\tilde{u}_\pm^{(0)}(s,\theta) = -\frac{\sin(s\theta)}{\mu_\pm s \cos(\pi s)} \langle \tilde{p} \rangle(s)$$
$$+ \left[\frac{\cos(s\theta)}{(\mu_+ + \mu_-) s \sin(\pi s)} + \frac{(\mu_+ - \mu_-)\sin(s\theta)}{2\mu_\pm(\mu_+ + \mu_-) s \cos(\pi s)} \right] [\![\tilde{p}]\!](s)$$



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$[\![f]\!] = f_+ - f_-$ jump of the function f across the line $x_2 = 0$

$\langle f \rangle = (f_+ + f_-)/2$ average of the function f across the line $x_2 = 0$

Preliminary results on the unperturbed problem $\varepsilon = 0$

Two-terms asymptotic expansion of the unperturbed solution:

- traction ahead of the crack tip:

$$\sigma^{(0)}(r,0) = \frac{K_{\text{III}}^{(0)}}{\sqrt{2\pi}} r^{-1/2} + \frac{A_{\text{III}}^{(0)}}{\sqrt{2\pi}} r^{1/2} + O(r^{3/2}), \quad r \to 0$$

- displacement jump across the crack faces (crack opening):

$$[\![u^{(0)}]\!](r) = \frac{\mu_+ + \mu_-}{\mu_+ \mu_-} \left(\frac{2K_{\text{III}}^{(0)}}{\sqrt{2\pi}} r^{1/2} - \frac{2A_{\text{III}}^{(0)}}{3\sqrt{2\pi}} r^{3/2} \right) + O(r^{5/2}), \quad r \to 0$$

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Integral representation for the constants:

$$K_{\mathrm{III}}^{(0)} = -\sqrt{\frac{2}{\pi}} \int_0^\infty \left\{ \langle p \rangle(-r) + \frac{\eta}{2} [\![p]\!](-r) \right\} r^{-1/2} dr \qquad \text{(SIF)}$$

$$A_{\mathrm{III}}^{(0)} = \sqrt{\frac{2}{\pi}} \int_0^\infty \left\{ \langle p \rangle(-r) + \frac{\eta}{2} [\![p]\!](-r) \right\} r^{-3/2} dr$$

$\eta = (\mu_- - \mu_+)/(\mu_+ + \mu_-)$ contrast parameter

Preliminary results on the unperturbed problem $\varepsilon = 0$

Displacement gradient $\nabla u_{\pm}^{(0)}$ at an arbitrary point $\boldsymbol{Y} = (d\cos\varphi, d\sin\varphi)$:

$$\left.\frac{\partial u_{\pm}^{(0)}}{\partial x_1}\right|_{\boldsymbol{Y}} = \frac{1}{\pi d}\int_{-\infty}^{0} \frac{dx_1}{2\cos\varphi - x_1/d - d/x_1}\left\{\frac{[\![p]\!](x_1)}{\mu_+ + \mu_-}\left[\sin^2\varphi - \frac{1}{2}\cos\varphi\left(\frac{d}{x_1} - \frac{x_1}{d}\right)\right] + \right.$$

$$\left. + \frac{2\langle p\rangle(x_1) + \eta[\![p]\!](x_1)}{2\mu_{\pm}}\left(\sqrt{\frac{-x_1}{d}}\sin\frac{\varphi}{2} + \sqrt{\frac{d}{-x_1}}\sin\frac{3\varphi}{2}\right)\right\},$$

$$\left.\frac{\partial u_{\pm}^{(0)}}{\partial x_2}\right|_{\boldsymbol{Y}} = -\frac{1}{\pi d}\int_{-\infty}^{0} \frac{dx_1}{2\cos\varphi - x_1/d - d/x_1}\left\{\frac{[\![p]\!](x_1)\sin\varphi}{\mu_+ + \mu_-}\left[\cos\varphi + \frac{1}{2}\left(\frac{d}{x_1} - \frac{x_1}{d}\right)\right] + \right.$$

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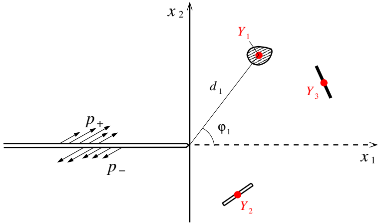
The point $\boldsymbol{Y}$ will be identified later with the centre of the defect.



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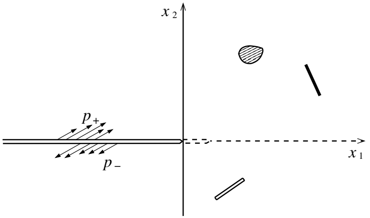
The structure of the asymptotic solution

The asymptotic solution is sought in the form:

$$u(\boldsymbol{x}, \varepsilon) = \; u^{(0)}(\boldsymbol{x}) \; + \; \varepsilon \sum_{j=1}^{3} W_j(\boldsymbol{\xi}_j) \; + \; \varepsilon^2 \sum_{j=1}^{3} u^{(j)}(\boldsymbol{x}) \; + \; \varepsilon^2 v(\boldsymbol{x}, \phi) \; + o(\varepsilon^2), \quad \varepsilon \to 0$$



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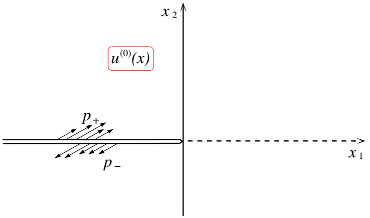

1. Solution of the unperturbed problem ($\varepsilon = 0$)

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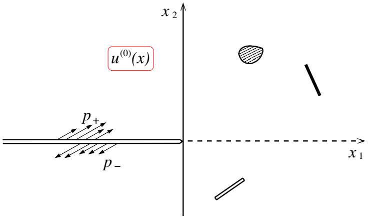

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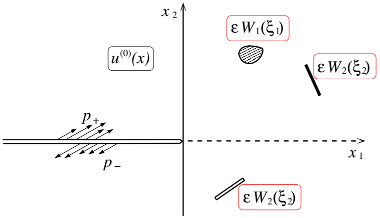

1. Solution of the unperturbed problem ($\varepsilon = 0$)
2. Boundary layers concentrated near the defects

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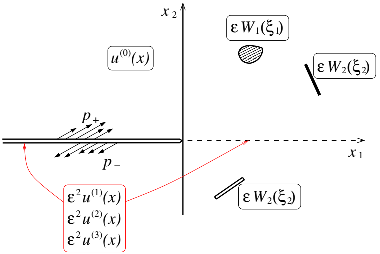

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2. Boundary layers concentrated near the defects
3. Additional terms to adjust the BC and IC disturbed by the boundary layers

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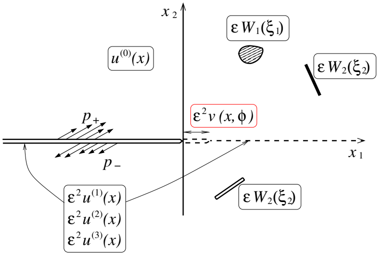

1. Solution of the unperturbed problem ($\varepsilon = 0$)
2. Boundary layers concentrated near the defects
3. Additional terms to adjust the BC and IC disturbed by the boundary layers
4. Perturbation associated with the crack advance $\varepsilon^2 \phi$

The structure of the asymptotic solution

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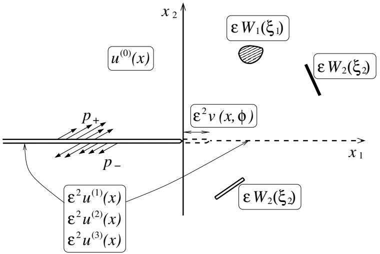

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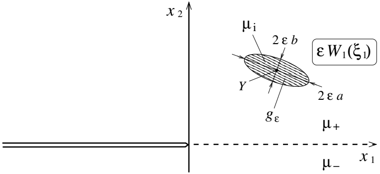
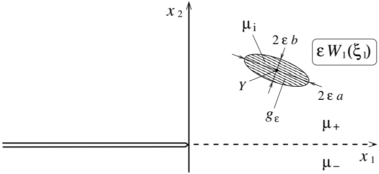
- Using the linearity of the problem, we analyse the perturbation of each defect separately (superposition principle).
- The method can be extended to a finite number of defects, provided that the distance between defects remains finite (composite is dilute).

Outline

1. Problem formulation
   - Problem formulation
   - Preliminary results on the unperturbed problem
2. The asymptotic procedure
   - The structure of the asymptotic solution
   - Regular perturbation and dipole fields generated by small defects
   - Singular perturbation for crack advance
   - Analysis of a stable quasi-static propagation
3. Illustrative examples
   - Shielding and amplification effects
   - Crack propagation and arrest

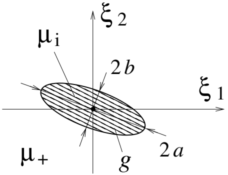
The boundary layer for an elliptic elastic inclusion

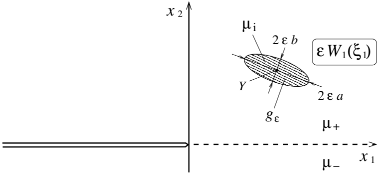




$\Rightarrow$

Scaled coordinates: $\boldsymbol{\xi} = \frac{\boldsymbol{x} - \boldsymbol{Y}}{\varepsilon}$





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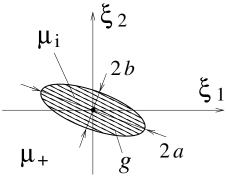


The BVP for the boundary layer $W(\boldsymbol{\xi})$:

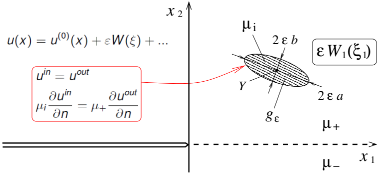
$$\Delta W^{in}(\boldsymbol{\xi}) = 0, \quad \boldsymbol{\xi} \in g, \quad \Delta W^{out} = 0, \quad \boldsymbol{\xi} \in \mathbb{R}^2 \setminus \overline{g}$$

$$W^{in} = W^{out} \quad \text{on} \quad \partial g$$

$$\mu_i \frac{\partial}{\partial \boldsymbol{n}} W^{in}(\boldsymbol{\xi}) - \mu_+ \frac{\partial}{\partial \boldsymbol{n}} W^{out}(\boldsymbol{\xi}) = (\mu_+ - \mu_i)\boldsymbol{n} \cdot \nabla u^{(0)}(\boldsymbol{Y}) + O(\varepsilon), \quad \varepsilon \to 0 \quad \text{on} \quad \partial g$$

$$W \to 0 \quad \text{as} \quad |\boldsymbol{\xi}| \to \infty$$

The boundary layer for an elliptic elastic inclusion



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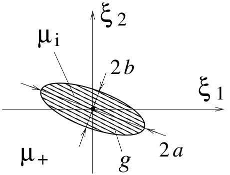

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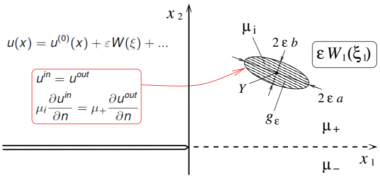

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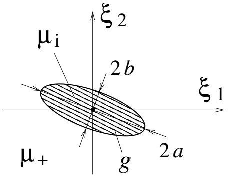

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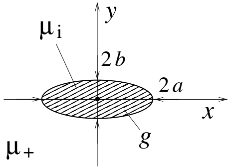
This problem can be reduced to a well-known problem[1]: *"Elliptic elastic inclusion in an infinite elastic plate subject to constant stresses at infinity"*

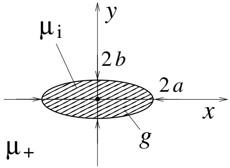
[1] N.J. Hardiman, QJMAM, 1954

The boundary layer for an elliptic elastic inclusion

Elliptic elastic inclusion in an infinite elastic plate subject to constant stresses at infinity



Elliptic elastic inclusion in an infinite elastic plate subject to constant stresses at infinity



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$$u^{out} = B_1 x + B_2 y + O(1) \quad \text{as} \quad r = \sqrt{x^2 + y^2} \to \infty$$

The boundary layer for an elliptic elastic inclusion

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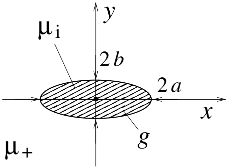
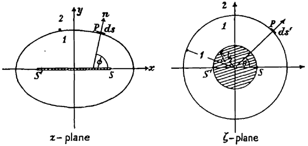


$\Rightarrow$

Conformal mapping + complex potentials:



$$z = c\left(\zeta + \frac{\lambda}{\zeta}\right)$$

$$c > 0, \quad 0 < \lambda < 1$$

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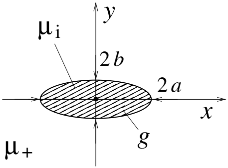

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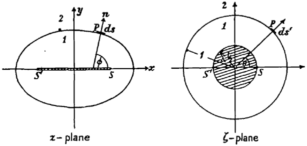

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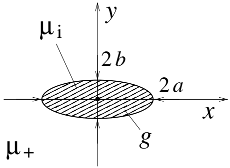
Solution:

$$u^{in} = A_1 x + A_2 y, \quad \text{in} \quad g$$

$$u^{out} = B_1 x + B_2 y - \frac{1}{2\pi}\{B_1, B_2\} \cdot \mathcal{M} \frac{\{x, y\}}{x^2 + y^2} + O(r^{-2}), \quad \text{as} \quad r \to \infty$$

$\mathcal{M}$ is a 2×2 matrix, it is called the dipole matrix

Elliptic elastic inclusion in an infinite elastic plate subject to constant stresses at infinity



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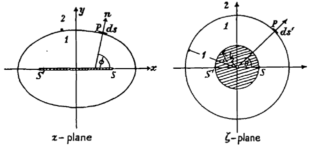

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The boundary layer for an elliptic elastic inclusion

Subtracting the linear term from the Hardiman solution we get the boundary layer:

$$W^{in}(\xi) = u^{in} - B_1 x - B_2 y = A_1 x + A_2 y - B_1 x + B_2 y, \quad \text{in} \quad g$$

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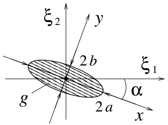

$$\begin{cases} x = \xi_1 \cos\alpha - \xi_2 \sin\alpha \\ y = \xi_1 \sin\alpha + \xi_2 \cos\alpha \end{cases}$$

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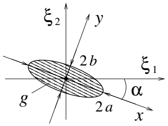

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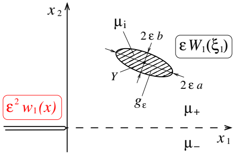

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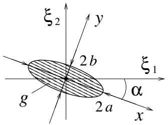

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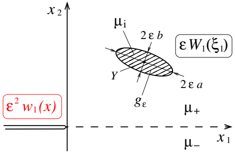

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Dipole field:

$$\varepsilon^2 w(\boldsymbol{x}) = -\frac{\varepsilon^2}{2\pi}\left[\nabla u^{(0)}\Big|_{\boldsymbol{Y}}\right] \cdot \left[\boldsymbol{\mathcal{M}} \frac{\boldsymbol{x} - \boldsymbol{Y}}{|\boldsymbol{x} - \boldsymbol{Y}|^2}\right] + o(\varepsilon^2), \quad \varepsilon \to 0$$

Dipole matrix:

$$\boldsymbol{\mathcal{M}} = -\frac{\pi}{2} ab(1+e)(\mu_\star - 1) \begin{bmatrix} \dfrac{1+\cos 2\alpha}{e+\mu_\star} + \dfrac{1-\cos 2\alpha}{1+e\mu_\star} & -\dfrac{(1-e)(\mu_\star - 1)\sin 2\alpha}{(e+\mu_\star)(1+e\mu_\star)} \\ -\dfrac{(1-e)(\mu_\star - 1)\sin 2\alpha}{(e+\mu_\star)(1+e\mu_\star)} & \dfrac{1-\cos 2\alpha_1}{e+\mu_\star} + \dfrac{1+\cos 2\alpha}{1+e\mu_\star} \end{bmatrix}$$

$$e = b/a, \quad \mu_\star = \mu_+/\mu_i$$

Dipole fields and dipole matrix for different types of defects

The dipole field for other type of defects can be found with a similar procedure:

$$\varepsilon^2 w(\boldsymbol{x}) = -\frac{\varepsilon^2}{2\pi} \left[\nabla u^{(0)} \Big|_{\boldsymbol{Y}} \right] \cdot \left[\mathcal{M} \frac{\boldsymbol{x} - \boldsymbol{Y}}{|\boldsymbol{x} - \boldsymbol{Y}|^2} \right] + o(\varepsilon^2), \quad \varepsilon \to 0$$

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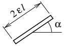
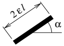
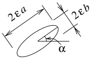
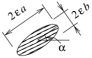
 microcrack	$\mathcal{M} = -\pi l^2 \begin{bmatrix} \sin^2\alpha & -\sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha & \cos^2\alpha \end{bmatrix}$
 rigid line inclusion	$\mathcal{M} = \pi l^2 \begin{bmatrix} \cos^2\alpha & \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha & \sin^2\alpha \end{bmatrix}$
 elliptic void	$\mathcal{M} = -\frac{\pi}{2}ab(1/e+1)\begin{bmatrix} 1-\cos 2\alpha + e(1+\cos 2\alpha) & -(1-e)\sin 2\alpha \\ -(1-e)\sin 2\alpha & 1+\cos 2\alpha + e(1-\cos 2\alpha) \end{bmatrix}$
 elliptic rigid inclusion	$\mathcal{M} = \frac{\pi}{2}ab(1/e+1)\begin{bmatrix} 1+\cos 2\alpha + e(1-\cos 2\alpha) & (1-e)\sin 2\alpha \\ (1-e)\sin 2\alpha & 1-\cos 2\alpha + e(1+\cos 2\alpha) \end{bmatrix}$

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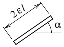
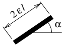
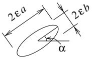
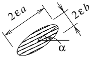
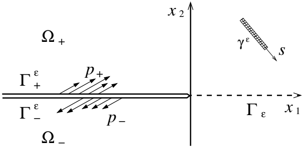
 microcrack	$\mathcal{M} = -\pi l^2 \begin{bmatrix} \sin^2\alpha & -\sin\alpha\cos\alpha \\ -\sin\alpha\cos\alpha & \cos^2\alpha \end{bmatrix}$
 rigid line inclusion	$\mathcal{M} = \pi l^2 \begin{bmatrix} \cos^2\alpha & \sin\alpha\cos\alpha \\ \sin\alpha\cos\alpha & \sin^2\alpha \end{bmatrix}$
 elliptic void	$\mathcal{M} = -\frac{\pi}{2} ab(1/e+1) \begin{bmatrix} 1 - \cos 2\alpha + e(1 + \cos 2\alpha) & -(1-e)\sin 2\alpha \\ -(1-e)\sin 2\alpha & 1 + \cos 2\alpha + e(1 - \cos 2\alpha) \end{bmatrix}$
 elliptic rigid inclusion	$\mathcal{M} = \frac{\pi}{2} ab(1/e+1) \begin{bmatrix} 1 + \cos 2\alpha + e(1 - \cos 2\alpha) & (1-e)\sin 2\alpha \\ (1-e)\sin 2\alpha & 1 - \cos 2\alpha + e(1 + \cos 2\alpha) \end{bmatrix}$

For an inclusion/void of general shape it is always possible to define an equivalent elliptic inclusion/void.

Dipole matrix for line defect with imperfect bonding



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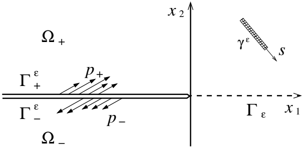

- Line defect with soft bonding (stiffness κ): $[\![\sigma]\!](s) = 0, \quad \sigma(s) = \kappa [\![u]\!](s)$

$$\begin{cases} \kappa = 0 & \Rightarrow \quad \sigma(s) = 0 & \text{microcrack} \\ \kappa = \infty & \Rightarrow \quad [\![u]\!](s) = 0 & \text{perfect bonding (no defect)} \end{cases}$$

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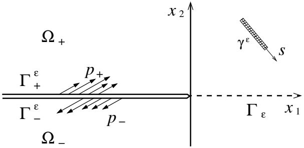

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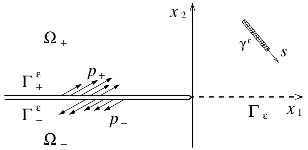
$$\begin{cases} \kappa = 0 & \Rightarrow \quad [\![\sigma]\!](s) = 0 & \text{no defect} \\ \kappa = \infty & \Rightarrow \quad \left.\dfrac{\partial^2 u}{\partial s^2}\right|_{\gamma^\epsilon} = 0 & \text{rigid line inclusion} \end{cases}$$



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$$\mathcal{M} = -\frac{\pi}{2}\frac{l^2}{\kappa l + 1}\begin{bmatrix} 1 - \cos 2\alpha & -\sin 2\alpha \\ -\sin 2\alpha & 1 + \cos 2\alpha \end{bmatrix}$$

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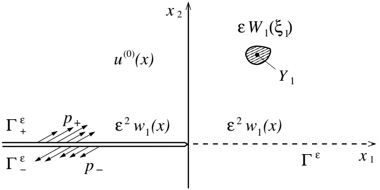
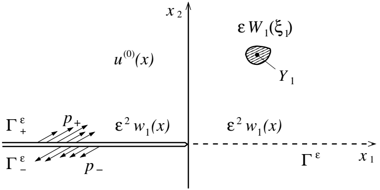


On the crack faces and on the interface:

$$u(\boldsymbol{x}) = u^{(0)}(\boldsymbol{x}) + \varepsilon^2 w_1(\boldsymbol{x})$$

“Effective” tractions and corresponding perturbation of SIF



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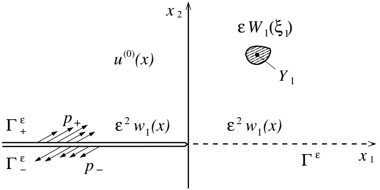

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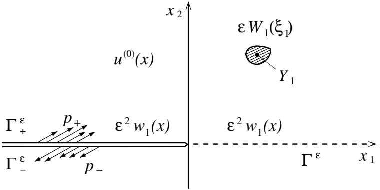

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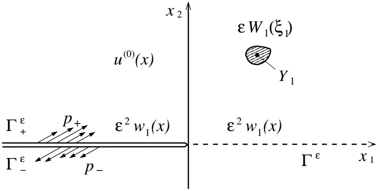

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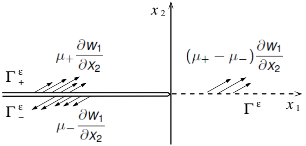
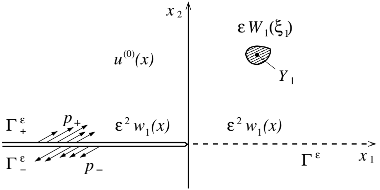
Problem for $u^{(1)}$:



"Effective" tractions and corresponding perturbation of SIF



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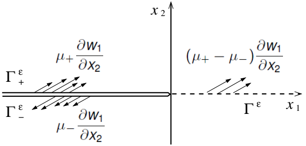

Corresponding SIF:

$$\Delta K_{\mathrm{III}}^{(1)} = -\sqrt{\frac{2}{\pi}} \int_{-\infty}^{0} \left(\langle \sigma^{(1)} \rangle + \frac{\eta}{2} [\![\sigma^{(1)}]\!] \right) (-x_1)^{-1/2} dx_1$$

$$= -\sqrt{\frac{2}{\pi}} \frac{\mu_+ \mu_-}{\mu_+ + \mu_-} \nabla u^{(0)}\Big|_{\boldsymbol{Y}_1} \cdot \boldsymbol{\mathcal{M}}_1 \boldsymbol{c}_1$$

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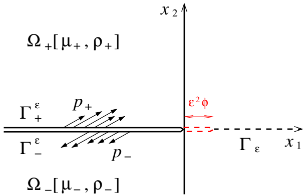
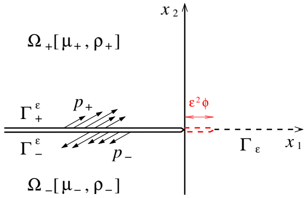
Singular perturbation for crack advance



[1] Piccolroaz et al., JMPS 2009



The Betti identity:

$$\int_{-\infty}^{\infty} \Big\{ [\![U]\!](x_1' - x_1)\langle\sigma\rangle(x_1) + \langle U\rangle(x_1' - x_1)[\![\sigma]\!](x_1) - \langle\Sigma\rangle(x_1' - x_1)[\![u]\!](x_1) \Big\} dx_1 = 0$$

u, σ physical solution

U, Σ weight functions

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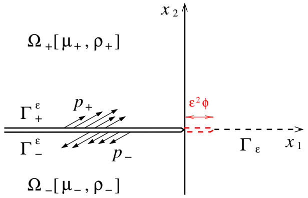

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u, σ physical solution $\qquad$ U, Σ weight functions

The weight functions[1]:

$$[\![U]\!](x_1) = \begin{cases} \dfrac{1-i}{\sqrt{2\pi}} x_1^{-1/2}, & x_1 > 0, \\ 0, & x_1 < 0, \end{cases} \qquad \langle U\rangle(x_1) = \eta/2[\![U]\!](x_1),$$

$$\langle\Sigma\rangle(x_1) = \begin{cases} 0, & x_1 > 0, \\ \dfrac{(1-i)\mu_+\mu_-}{2\sqrt{2\pi}(\mu_+ + \mu_-)}(-x_1)^{-3/2}, & x_1 < 0. \end{cases}$$

The weight functions are special singular solution of the homogeneous problem (traction-free crack faces).

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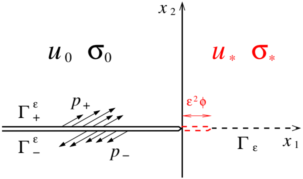
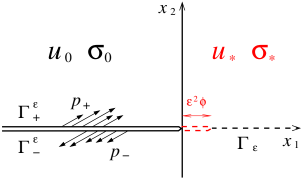
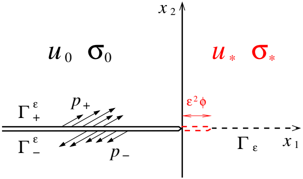

x_2
u_0 σ_0
u_* σ_*
$\varepsilon^2\phi$
Γ_+^ε
p_+
Γ_-^ε
p_-
Γ_ε
x_1



Write the Betti identity for the unperturbed solution and for the perturbed solution, subtract one from the other and apply Fourier transform:

$$[\![\overline{U}]\!]^+(\overline{\sigma}_0^+ - e^{i\beta\varepsilon^2\phi}\overline{\sigma}_\star^+) - \overline{\Sigma}^-([\![\overline{u}_0]\!]^- - e^{i\beta\varepsilon^2\phi}[\![\overline{u}_\star]\!]^-) = 0$$



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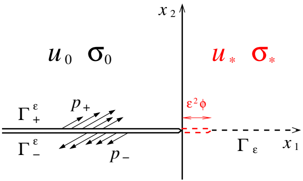
Two terms asymptotics:

$$\overline{\sigma}_0^+ = \frac{(1+i)K_{\mathrm{III}}^{(0)}}{2}\beta_+^{-1/2} - \frac{(1-i)A_{\mathrm{III}}^{(0)}}{4}\beta_+^{-3/2} + O(\beta_+^{-5/2}), \quad \beta_\pm \to \infty$$

$$[\![\overline{u}_0]\!]^- = -\frac{(1+i)(\mu_+ + \mu_-)K_{\mathrm{III}}^{(0)}}{2\mu_+\mu_-}\beta_-^{-3/2} + \frac{(1-i)(\mu_+ + \mu_-)A_{\mathrm{III}}^{(0)}}{4\mu_+\mu_-}\beta_-^{-5/2} + O(\beta_-^{-7/2}), \quad \beta_\pm \to \infty$$



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Substituting into the Betti identity and collecting like powers of $\beta_\pm$:

$$\left\{\frac{1+i}{2}(K_{\text{III}}^{(0)} - K_{\text{III}}^\star) + \frac{1-i}{4}i\varepsilon^2\phi A_{\text{III}}^\star\right\}(\beta_+^{-1} - \beta_-^{-1}) + O(\beta^{-2}) = 0$$

$$K_{\text{III}}^* - K_{\text{III}}^{(0)} := \varepsilon^2\Delta K_{\text{III}}^\phi = \frac{\varepsilon^2\phi}{2}A_{\text{III}}^{(0)}$$

$$A_{\text{III}}^{(0)} = \sqrt{\frac{2}{\pi}}\int_{-\infty}^0 \left(\langle p\rangle(x_1) + \frac{\eta}{2}[\![p]\!](x_1)\right)(-x_1)^{-3/2}dx_1$$

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Analysis of a stable quasi-static propagation

The stress intensity factor is expanded as follows:

$$K_{\text{III}} = K_{\text{III}}^{(0)} + \varepsilon^2 \left(\Delta K_{\text{III}}^{\phi} + \sum_{j=1}^{3} \Delta K_{\text{III}}^{(j)} \right) + o(\varepsilon^2), \quad \varepsilon \to 0$$

$\Delta K_{\text{III}}^{\phi} = \dfrac{\varepsilon^2 \phi}{2} A_{\text{III}}^{(0)}$: perturbation produced by the elongation of the crack along the interface

$\sum_{j=1}^{3} \Delta K_{\text{III}}^{(j)}$: perturbation produced by the defects

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We assume that the crack propagation is stable and quasi-static ($G = G_c$): $\Delta G = 0$

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$$G = \frac{1}{4}\left(\frac{1}{\mu_+} + \frac{1}{\mu_-}\right) K_{\text{III}}^2 \quad \Rightarrow \quad \Delta K_{\text{III}}^{\phi} + \sum_{j=1}^{3} \Delta K_{\text{III}}^{(j)} = 0 \quad \Rightarrow \quad \boxed{\phi = -\frac{2}{A_{\text{III}}^{(0)}} \sum_{j=1}^{3} \Delta K_{\text{III}}^{(j)}}$$

$$\Delta K_{\text{III}}^{(j)} = -\sqrt{\frac{2}{\pi}} \frac{\mu_+ \mu_-}{\mu_+ + \mu_-} \nabla u^{(0)} \Big|_{\boldsymbol{Y}_j} \cdot \boldsymbol{\mathcal{M}}_j \boldsymbol{c}_j, \qquad \boldsymbol{c}_j = \frac{1}{2 d_j^{3/2}} \left[-\sin\frac{3\varphi_j}{2}, \cos\frac{3\varphi_j}{2} \right]$$

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Shielding and amplification effects

Given the geometry and position of the defect:

$$\Delta K_{\text{III}} = -\sqrt{\frac{2}{\pi}} \frac{\mu_+ \mu_-}{\mu_+ + \mu_-} \left. \nabla u^{(0)} \right|_{\boldsymbol{Y}} \cdot \boldsymbol{\mathcal{M}} \boldsymbol{c}$$

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Definition:

$\Delta K_{\mathrm{III}} < 0$:	*shielding effect*	"the defect is *preventing* the propagation"
$\Delta K_{\mathrm{III}} > 0$:	*amplification effect*	"the defect is *promoting* the propagation"
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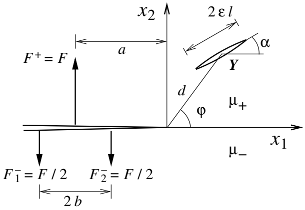
Example: shielding/amplification diagrams for macro-microcrack interaction



$d = 1$, $2\ \varepsilon l = 0.02$, $a = 2$

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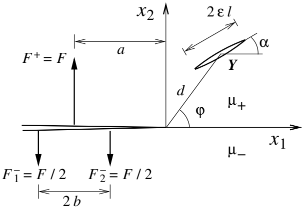

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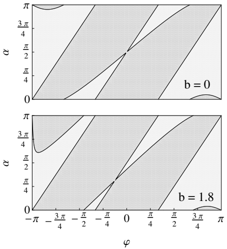
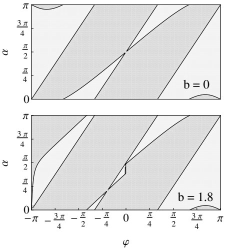
Homogeneous plane:



$\mu_+ = \mu_-$

Bimaterial plane:



$\mu_+/\mu_- = 0.1$

Outline

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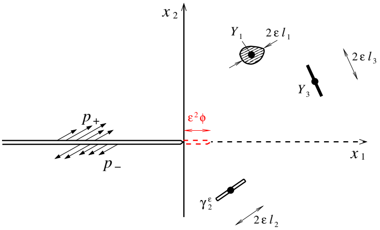
Crack propagation and arrest



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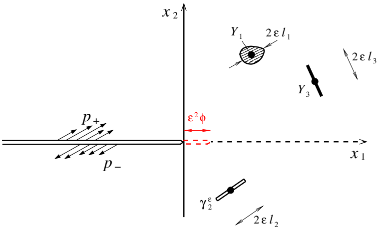
$$\phi = -\frac{2}{A_{\text{III}}^{(0)}} \sum_{j=1}^{3} \Delta K_{\text{III}}^{(j)}$$



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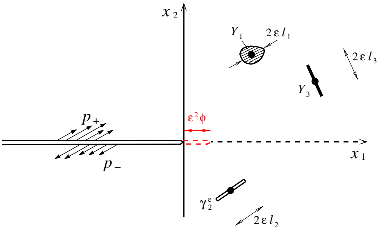
- the crack "arrests" when a neutral configuration is reached ($\phi = 0$)

Crack propagation and arrest



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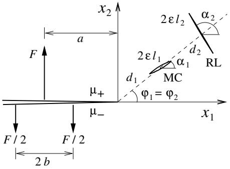
The total crack elongation is computed as:

$$x(N) = \sum_{i=0}^{N} \varepsilon^2 \phi_i$$

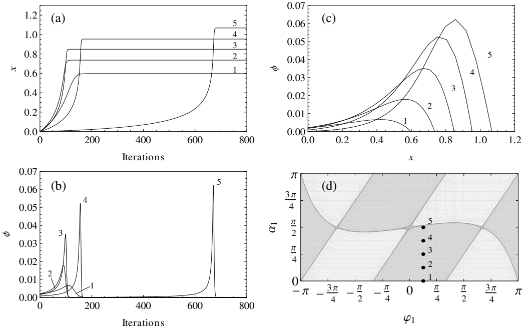
where N is the number of iterations.

Main crack interacting with a microcrack (MC) and a rigid line inclusion (RL):

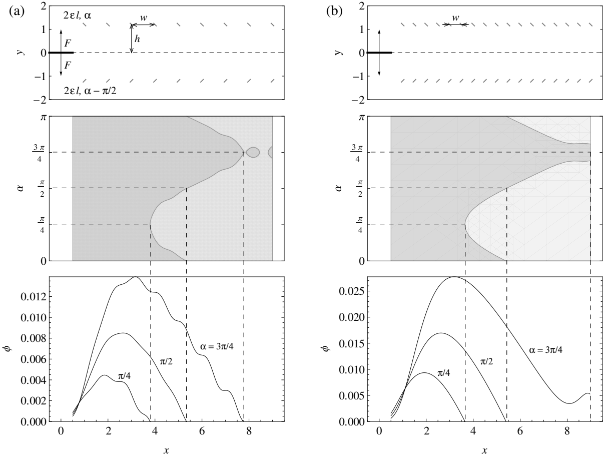


$$d_2 = 2d_1,\ l_2 = 2l_1,\ \varphi_2 = \varphi_1,\ \alpha_2 = \pi/2 - \alpha_1$$

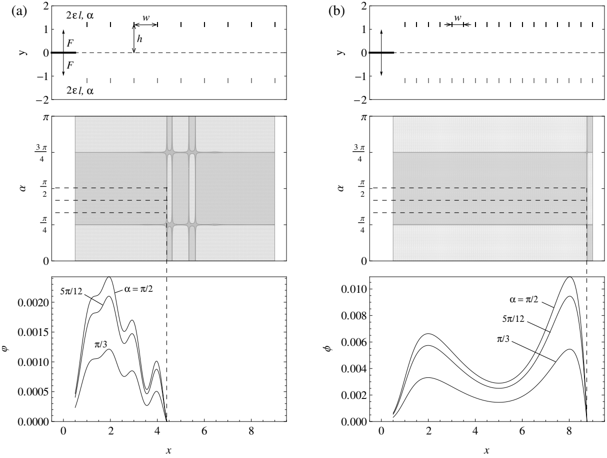
Crack propagation and arrest ($\varphi_1 = \pi/8$, $\alpha_1 = 0, \pi/8, \pi/4, 3\pi/8, \pi/2$):



Crack propagation in an array of defects - I



Crack propagation in an array of defects - II



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$$w(\boldsymbol{x}) = -\frac{1}{2\pi}\left[\nabla u^{(0)}\Big|_{\boldsymbol{Y}}\right] \cdot \left[\boldsymbol{\mathcal{M}}\frac{\boldsymbol{x}-\boldsymbol{Y}}{|\boldsymbol{x}-\boldsymbol{Y}|^2}\right]$$

The small defect is replaced by "effective" tractions on the crack faces giving the same perturbation:

$$\langle\sigma\rangle = -\frac{1}{2}(\mu_+ + \mu_-)\frac{\partial w}{\partial x_2}, \quad [\![\sigma]\!] = -(\mu_+ - \mu_-)\frac{\partial w}{\partial x_2}$$

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3. Asymptotic formula for the crack advance:

$$\phi = -\frac{2}{A_{\text{III}}^{(0)}}\sum_{j=1}^{3}\Delta K_{\text{III}}^{(j)}$$

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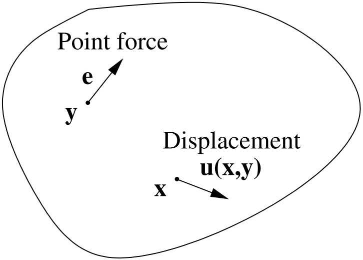
Submitted.

Symmetric and skew-symmetric weight functions in linear fracture mechanics

Green's function

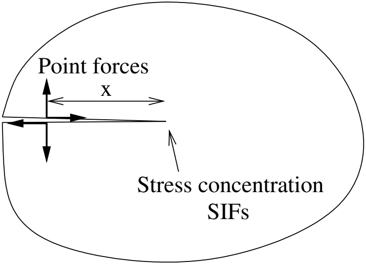
Solution of the displacement field at point $\boldsymbol{x}$ produced by a unit concentrated body force $\boldsymbol{e}$ located at point $\boldsymbol{y}$.



$$u_i(\boldsymbol{x},\boldsymbol{y}) = G_{ij}(\boldsymbol{x},\boldsymbol{y})e_j$$

$$G_{ij}(\boldsymbol{x},\boldsymbol{y}) = \frac{1}{16\pi\mu(1-\nu)r}\left[(3-4\nu)\delta_{ij} + r_{,i}\, r_{,j}\right], \quad \boldsymbol{r} = \boldsymbol{x} - \boldsymbol{y}, \quad r = |\boldsymbol{r}|$$

Fundamental solutions in linear fracture mechanics



$$K = \sqrt{\frac{2}{\pi}}(1+i)x^{-1/2}$$

$$K_{\text{I}} = \sqrt{\frac{2}{\pi}}x^{-1/2}, \quad K_{\text{II}} = \sqrt{\frac{2}{\pi}}x^{-1/2}$$

Fundamental solutions in linear fracture mechanics are known as weight functions.

Definition of Weight Functions (Bueckner, 1970)

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SIF associated with concentrated point forces applied on the crack faces.

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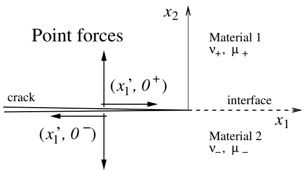
2D case



2D weight function: $w = w(x_1')$

$$K = \int_{-\infty}^{0} w(x_1') \underbrace{p(x_1')}_{\text{distributed load}} dx_1'$$

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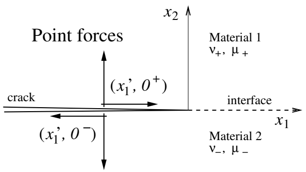

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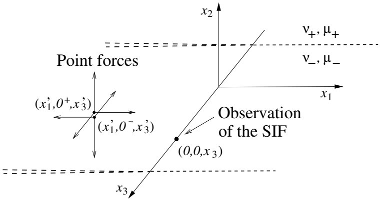

3D weight function: $w = w(x_1', x_3', x_3)$

$$K(x_3) = \int_{-\infty}^{\infty} \int_{-\infty}^{0} w(x_1', x_3', x_3) \underbrace{p(x_1', x_3')}_{\text{distributed load}} dx_1' dx_3'$$

Definition

Singular solution to the elastic crack problem with homogeneous boundary conditions (traction-free crack faces): displacement field $\boldsymbol{U}(x_1, x_2)$, stress field $\boldsymbol{\Sigma}(x_1, x_2)$.

Generalization of Weight Functions (Willis-Movchan, 1995)

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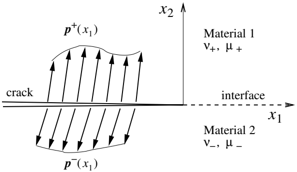

Model domain for the **physical solution**
The crack lies on the left-hand side

$$\boldsymbol{u} \sim \sum_{j=1}^{3} K_j r^{1/2+i\epsilon} f_j(\theta)$$

$$\boldsymbol{\sigma} \sim \sum_{j=1}^{3} K_j r^{-1/2+i\epsilon} g_j(\theta)$$

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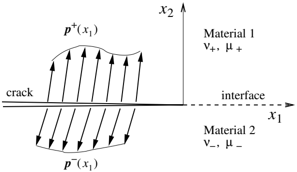

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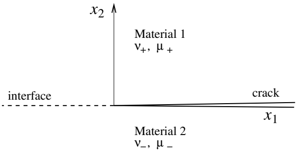

Model domain for the singular solutions
The crack lies on the right-hand side

$$\boldsymbol{U} \sim \sum_{j=1}^{3} K_j r^{-1/2+i\epsilon} F_j(\theta)$$

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- derivation of SIFs associated to concentrated forces on the crack faces (Bueckner weight functions):

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- derivation of SIFs associated to a general asymmetrical loading on the crack faces:

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- solution of perturbation problems (load and/or geometrical perturbations)