HELE-SHAW FLOW WITH A SMALL OBSTACLE

Sergei V.Rogosin[1,†], Gennady Mishuris[2]

[1] Department of Economics, Belarusian State University
Nezavisimosti ave 4, 220030 Minsk, Belarus
e-mail: rogosin@bsu.by

[2] Department of Mathematical and Physical Sciences, Aberystwyth University
Ceredegion SY23 3BZ, Aberystwyth, UK
e-mail: ggm@aber.ac.uk

[†] Lecturer

International Conference
"Topics of Operator Theory" (STOP-2011),
Instituto Superior Téchnico, Lisboa, Portugal,

June 23–29, 2011

S.R. is partially supported by International Travel Grants - 2010/R2 No. 45239 Travel for Collaboration granted by the Royal Society, and by "Belarusian Fund for Fundamental Scientific Research", G.M. is thankful for support from the FP7-PEOPLE-2009-IAPP grant PIAP-GA-2009-251475 HYDROFRAC.

The aim:

♡ to provide an asymptotic analysis of the Hele-Shaw moving boundary value problem with a small obstacle in the flow.

Plan:

♠ historical remarks;

♠ geometry corresponding to the considered problem;

♠ real-variable Hele-Shaw model in a domain with an obstacle;

♠ Maz'ya & Movchan asymptotic representation of Green's function;

♠ asymptotic study of the solution to the Hele-Shaw boundary value problem in a domain with an obstacle.

Few historical remarks.

- H. S. Hele-Shaw (1898) - description of an experiment in 2D cell
- S. L. Leibenson (1932) - application in oil production
- Yu. P. Vinogradov, P. P. Kufarev (1948) - complex Hele-Shaw model
- S. Richardson (1972) - rediscovering of complex Hele-Shaw model
- P. Ya. Polubarinova-Kochina (1945), L. A. Galin (1945) - first exact solutions to Hele-Shaw problem
- B. Gustafsson (1984) - local existence of rational solution
- B. Gustafsson (1984) - differential equation for Hele-Shaw model
- B. Gustafsson (1985) - existence of weak solution
- M. Reissig, L. von Wolfersdorf (1993) - new proof of local existence
- P. G. Saffman (1986), S. Tanveer (1993) - surface tension regularization
- L. Romero (1981) - kinetic undercooling regularization
- J. Escher & G.Simonett (1997) - maximum regularity
- S.N. Antontsev, C.R. Gonçalves, A.M. Meirmanov (2003) - existence and uniqueness of the classical solution

Description of the geometry.

Viscous incompressible fluid occupies a doubly connected domain $D_1(t)$ at a time instant $t \geq 0$.

Internal domain F is a fixed small obstacle (hole).

The simply connected domain without hole will be denoted $D(t)$.

It is supposed that $F \subset D(0)$ has a nonempty interior, and the diameter of obstacles $\delta := diam\, F$ is positive.

$D(0)$ is supposed to be open bounded set with a smooth boundary and

$$dist\ \{\partial\, F, \partial\, D(0)\} = 2d > 0. \tag{1}$$

Without loss of generality we can assume that δ and d ($\delta < d$) are dimensionless parameters and that $diam\, D(0) = 1$.

Derivation of the model.

Two-dimensional potential flow of incompressible fluid in the Hele-Shaw cell, i.e. in a gap between two parallel plates of distance h.

The flow is modelled by the velocity field $\mathbf{V} = (V_1, V_2, V_3)$ in $D_1(t) \in \mathbb{R}^2$:

$$\frac{\partial \mathbf{V}}{\partial t} = 0, \; V_3 = 0. \tag{2}$$

$\mathbf{V} = (V_1, V_2)$ is proportional to the pressure p gradient:

$$\mathbf{V} = -\frac{h^2}{12\mu} \nabla p, \tag{3}$$

where μ is the viscosity coefficient of the fluid. For h and μ being constant

$$\triangle p = 0. \tag{4}$$

$$p(z,t) \sim -\frac{Q(t)}{2\pi} \log |z - z_0|, \quad |z| \to z_0. \tag{5}$$

$$\frac{\partial p}{\partial n} = 0, \; z \in \partial F, \tag{6}$$

Surface tension dynamic boundary condition

$$p(z) = 0, \quad z \in \Gamma(t) = \partial D(t), \tag{7}$$

and kinematic boundary condition

$$\frac{d\Gamma}{dt} = \mathbf{V}, \quad z \in \Gamma(t) = \partial D(t). \tag{8}$$

Together with Hele-Shaw equation it gives

$$\frac{d\Gamma}{dt} = -\frac{h^2}{12\mu}\nabla p. \tag{9}$$

Real-variable Hele-Shaw model.

New unknown one-parametric family of $\mathcal{C}^2$-diffeomorphisms

$$w(s,t) = (u(s,t), v(s,t)) : \partial\,\mathbb{U}\times I \to \Gamma(t), \quad \mathbb{U} = \{s = (s_1, s_2) \in \mathbb{R}^2 : |z| < 1\}. \tag{10}$$

The function $w(s,t)$ in (10) determines an unknown parametrization of the free boundary $\Gamma(t)$

(i) $w(s,t) \in \Gamma(t)$ for all $(s,t) \in \partial\,\mathbb{U} \times I$,

(ii) $w(\cdot,t) : \partial\,\mathbb{U} \to \Gamma(t)$ is a $\mathcal{C}^2$-diffeomorphism for each fixed $t \in I$,

(iii) $w(\cdot,\cdot) \in \mathcal{C}^2\left(\partial\,\mathbb{U} \times I; \mathbb{R}^2\right)$.

It follows from the relations (4)–(7) that for each fixed appropriate t the unknown pressure p coincides up to constant factor with Green's function of the operator $-\triangle$ in the doubly connected domain $D_1(t)$ with the homogeneous Neumann data on the fixed boundary ∂F and the homogeneous Dirichlet data on the free boundary $\Gamma(t)$.

$$p = Q \cdot \mathcal{G}_{D_1(t)}, \tag{11}$$

and $\mathcal{G}_{D_1(t)}$ is the solution of the following mixed boundary value problem

$$\triangle \mathcal{G}_{D_1(t)}(z, z_0) + \delta_0(z - z_0) = 0, \quad z \in D_1(t), \tag{12}$$

$$\mathcal{G}_{D_1(t)}(z, z_0) = 0, \quad z \in \Gamma(t), \tag{13}$$

$$\frac{\partial \mathcal{G}_{D_1(t)}}{\partial n}(z, z_0) = 0, \quad z \in \partial F. \tag{14}$$

Real-variable formulation in terms of diffeomorphisms.

Problem ($\mathbf{HS_0}$). *Find a pair* $\{w(s,t); \mathcal{G}(z,z_0;t)\}$, *such that* $w(s,t)$: $\partial\,\mathbb{U} \times I \to \mathbb{R}^2$ *is a* $\mathcal{C}^2$*-diffeomorphism satisfying*

(i) $w(s,t) \in \Gamma(t)$ *for all* $(s,t) \in \partial\,\mathbb{U} \times I$*;*

(ii) $w(\cdot,t) : \partial\,\mathbb{U} \to \Gamma(t)$ *is a* $\mathcal{C}^2$*-diffeomorphism for each fixed* $t \in I$*;*

(iii) $w^{(0)}(s) = w(s,0)$ *is a given* $\mathcal{C}^2$*-diffeomorphism of the unit circle* $\partial\,\mathbb{U}$*, which describes the boundary* $\Gamma(0)$ *of initial domain* $D_1(0)$*;*

(iv) $\mathcal{G}(z,z_0;t)$ *is Green's function of the operator* $-\triangle$ *in the doubly connected domain* $D_1(t)$ *with the homogeneous Neumann data on the fixed boundary* $\partial\,F$ *and the homogeneous Dirichlet data on the free boundary* $\Gamma(t)$*;*

(v) $\partial_t\, w(s,t) = -\frac{Qh^2}{12\mu} \cdot \nabla\mathcal{G}(w(s,t), z_0; t)$ *for all* $(s,t) \in \partial\,\mathbb{U} \times I$*.*

Mazya-Movchan asymptotic results.

Let $\Omega \subset \mathbb{R}^2$ be a bounded open plane domain with compact closure $cl\,\Omega$ and a smooth boundary $\partial\,\Omega$. Let ω be a bounded domain in $\mathbb{R}^2$ with compact closure $cl\,\omega$ and smooth boundary $\partial\,\omega$. Let the origin O belongs to the intersection of Ω and ω, i.e. $O \in \Omega \cap \omega$. $\omega_\varepsilon = \{z \in \mathbb{R}^2 : \varepsilon^{-1}(z - O) \in \omega\}$, where ε is a small positive parameter, such that $\omega_\varepsilon \subset \Omega$. Let Ω_ε be an open doubly connected domain $\Omega_\varepsilon = \Omega \setminus cl\,\omega_\varepsilon$.

Denote by $\mathcal{G}_\varepsilon$ Green's function of the operator $-\triangle$ with the Dirichlet data on $\partial\,\Omega$ and the Neumann data on ω_ε, namely

$$\triangle_z\,\mathcal{G}_\varepsilon(z,\zeta) + \delta(z-\zeta) = 0, \quad z,\zeta \in \Omega_\varepsilon, \tag{15}$$

$$\mathcal{G}_\varepsilon(z,\zeta) = 0, \quad z \in \partial\,\Omega, \zeta \in \Omega_\varepsilon, \tag{16}$$

$$\frac{\partial\,\mathcal{G}_\varepsilon}{\partial\,n_z}(z,\zeta) = 0, \quad z \in \partial\,\omega_\varepsilon, \zeta \in \Omega_\varepsilon. \tag{17}$$

Let $\mathcal{G}$ be Green's function of the Dirichlet problem for operator $-\triangle$ in Ω

$$\mathcal{G}(z,\zeta) = \frac{1}{2\pi} \log \frac{1}{|z-\zeta|} - \mathcal{H}(z,\zeta), \tag{18}$$

where $\mathcal{H}$ is a unique solution to the following problem

$$\triangle_z \mathcal{H}(z,\zeta) = 0, \quad z,\zeta \in \Omega, \tag{19}$$

$$\mathcal{H}(z,\zeta) = \frac{1}{2\pi} \log \frac{1}{|z-\zeta|}, \quad z \in \partial\Omega, \zeta \in \Omega. \tag{20}$$

Introduce auxiliary variables (scaled coordinate) $\xi = \frac{1}{\varepsilon} z, \eta = \frac{1}{\varepsilon}\zeta$.

Define the Neumann function $\mathcal{N}$ of the exterior Neumann problem for operator $-\triangle$ in the domain $\mathbb{R}^2 \setminus \omega$

$$\mathcal{N}(\xi, \eta) = \frac{1}{2\pi} \log \frac{1}{|\xi - \eta|} - h_N(\xi, \eta), \tag{21}$$

where h_N is a unique solution to the following problem (regular part of the Neumann function)

$$\triangle_\xi \, h_N(\xi, \eta) = 0, \quad \xi, \eta \in \mathbb{R}^2 \setminus cl\, \omega, \tag{22}$$

$$\frac{\partial \, h_N}{\partial \, n_\xi}(\xi, \eta) = \frac{1}{2\pi} \frac{\partial}{\partial \, n_\xi} \log \frac{1}{|\xi - \eta|}, \quad \xi \in \partial \, \omega, \eta \in \mathbb{R}^2 \setminus cl\, \omega, \tag{23}$$

$$h_N(\xi, \eta) \to 0, \quad \text{as } |\xi| \to \infty, \quad \eta \in \mathbb{R}^2 \setminus cl\, \omega. \tag{24}$$

Introduce the vector dipole fields $\mathcal{D}(\xi) = (\mathcal{D}_1(\xi), \mathcal{D}_2(\xi))^T$:

$$\triangle \mathcal{D}(\xi) = 0, \quad \xi \in \mathbb{R}^2 \setminus cl\, \omega, \tag{25}$$

$$\frac{\partial \mathcal{D}_j}{\partial n}(\xi) = n_j, \quad \xi \in \partial \omega, \ j = 1, 2, \tag{26}$$

$$\mathcal{D}_j(\xi) \to 0, \quad \text{as } |\xi| \to \infty, \quad j = 1, 2, \tag{27}$$

where n_1, n_2 are components of the inward unit normal vector to $\partial \omega$. The dipole fields $\mathcal{D}_j$, $j = 1, 2$, satisfy the asymptotic representation for sufficiently large $|\xi|$

$$\mathcal{D}_j(\xi) = \frac{1}{2\pi} \sum_{k=1}^{2} \frac{\mathcal{P}_{jk} \xi_k}{|\xi|^2} + O(|\xi|^{-2}), \tag{28}$$

where

$$\mathcal{P}_{jk} = -\delta_{jk} \cdot meas\,(\omega) - \int\limits_{\mathbb{R}^2 \setminus \omega} \nabla \mathcal{D}_j(\xi) \cdot \nabla \mathcal{D}_k(\xi) d\xi. \tag{29}$$

(**Mazya-Movchan, 2009**) *Green's function* $\mathcal{G}_\varepsilon$ *for the boundary value problem (15)–(17) with the Dirichlet data on* $\partial\,\Omega$ *and the Neumann data on* $\partial\,\omega_\varepsilon$ *has the asymptotic representation*

$$\mathcal{G}_\varepsilon(z,\zeta) = \mathcal{G}(z,\zeta) + \mathcal{G}^*_\varepsilon(z,\zeta) + r_\varepsilon(z,\zeta), \tag{30}$$

where $\mathcal{G}$ *is Green's function of the Dirichlet problem for Laplace equation in the domain* Ω*, the principal part* $\mathcal{G}^*_\varepsilon$ *of the asymptotic representation is defined in terms of Neumann function* $\mathcal{N}$*, regular part* $\mathcal{H}$ *of Green's function* $\mathcal{G}$ *and the dipole field* $\mathcal{D}$

$$\mathcal{G}^*_\varepsilon(z,\zeta) = \mathcal{N}\left(\frac{z}{\varepsilon},\frac{\zeta}{\varepsilon}\right) + \frac{1}{2\pi}\log\left(\frac{|z-\zeta|}{\varepsilon}\right) + \varepsilon\mathcal{D}\left(\frac{z}{\varepsilon}\right)\cdot\nabla_z\mathcal{H}(0,\zeta) + \varepsilon\mathcal{D}\left(\frac{\zeta}{\varepsilon}\right)\cdot\nabla_\zeta\mathcal{H}(z,0), \tag{31}$$

and the remainder satisfies the following uniform inequality

$$|r_\varepsilon(z,\zeta)| \leq const\cdot\varepsilon^2. \tag{32}$$

Corollaries:

1. (**Mazya-Movchan, 2009**) *Let* min $\{|z|,|\zeta|\} > 2\varepsilon$. *Then the following asymptotic formula holds:*

$$\mathcal{G}_\varepsilon(z,\zeta) = \mathcal{G}(z,\zeta) - \frac{\varepsilon^2}{4\pi^2}\frac{z^T}{|z|^2}\mathcal{P}\frac{\zeta}{|\zeta|^2} + \tag{33}$$

$$+\frac{\varepsilon^2}{2\pi}\left\{\frac{z^T}{|z|^2}\mathcal{P}\nabla_z\mathcal{H}(0,\zeta) + \frac{\zeta^T}{|\zeta|^2}\mathcal{P}\nabla_\zeta\mathcal{H}(z,0)\right\}+$$

$$+\varepsilon^2 O\left(\frac{1}{|z|^2}+\frac{1}{|\zeta|^2}\right),$$

where $\mathcal{H}$ *is the regular part of Green's function* $\mathcal{G}$ *in the domain* Ω *defined by (19)-(20), and* $\mathcal{P}$ *is the dipole matrix for* ω *defined in* (29).

2. (**Mazya-Movchan, 2009**) *For all* $z, \zeta \in \Omega_\varepsilon$ *asymptotic formula (30) can be presented in the following form:*

$$\mathcal{G}_\varepsilon(z,\zeta) = \frac{1}{2\pi} \log \frac{1}{|z-\zeta|} - h_N\left(\frac{1}{\varepsilon}z, \frac{1}{\varepsilon}\zeta\right) - \mathcal{H}(0,0) - \tag{34}$$

$$-\left(z - \varepsilon\,\mathcal{D}\left(\frac{1}{\varepsilon}z\right)\right) \cdot \nabla_z \mathcal{H}(0,\zeta) - \left(\zeta - \varepsilon\,\mathcal{D}\left(\frac{1}{\varepsilon}\zeta\right)\right) \cdot \nabla_\zeta \mathcal{H}(z,0) +$$

$$+ O\left(\varepsilon^2 + |z|^2 + |\zeta|^2\right).$$

Representation of the solution.

$$F = \omega_\varepsilon, \quad D(t) = \Omega. \tag{35}$$

Thus

$$\mathcal{G}(z, z_0; t) = \mathcal{G}_\varepsilon(z, z_0). \tag{36}$$

This function have to satisfy the following relations:

$$\partial_t\, w(s,t) = -\frac{Qh^2}{12\mu} \cdot \nabla_z \mathcal{G}_\varepsilon(w(s,t), z_0) \quad \text{for all} \quad (s,t) \in \partial\, \mathbb{U} \times I, \tag{37}$$

$$w(s,0) = w^{(0)}(s), \quad s \in \partial\, \mathbb{U}, \tag{38}$$

where $w^{(0)}(s) \in \mathcal{C}^2(\partial\, \mathbb{U})$ is a given function.

We consider four essentially different situations:

(a) the source z_0 and ALL points $z = w(s,t)$ of the boundary $\partial\Gamma(0)$ of the initial domain $D_1(0)$ are distant from the boundary $\partial\omega_\varepsilon$ of the obstacle;

(b) the source z_0 is close to the boundary $\partial\omega_\varepsilon$ of the obstacle, but ALL points $z = w(s,t)$ of $\partial\Gamma(0)$ are distant from $\partial\omega_\varepsilon$;

(c) the source z_0 and SOME points $z = w(s,t)$ of $\partial\Gamma(0)$ are close to the boundary $\partial\omega_\varepsilon$ of the obstacle; we consider those points $z = w(s,t)$ which are distant from $\partial\omega_\varepsilon$;

(d) the source z_0 and SOME points $z = w(s,t)$ of $\partial\Gamma(0)$ are close to the boundary $\partial\omega_\varepsilon$ of the obstacle; we consider those points $z = w(s,t)$ which are close to $\partial\omega_\varepsilon$.