Speed Equation in Problems of Hydraulic Fracturing: Theory and Applications

Alexander M. Linkov

Institute for Problems of Mechanical Engineering
(Russian Academy of Sciences)
presently: Rzeszow University of Technology

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Hydraulic Fracturing

Essence and Brief Historical Overview

Drastic increase of the surface to which oil flows to the well

**ESSENCE**

1896 USA Patent No 556 669 pumping fluid under pressure to force acid further into rock; 1930s Dow Chemical Company discovered that fluid pressure could be applied to crack and deform rock leading to better well stimulation; 1947 First hydraulic treatment to stimulate well production in order to compare with the current technology (Kansas, Hugoton field)
Hydraulic Fracturing

Modern Applications

Today, hydraulic fracturing is used extensively in the petroleum industry to stimulate oil and gas wells in order to increase their productivity.

Hydraulic fracturing is also used to

- Increase heat production from geothermal reservoirs
- Measure in-situ stresses
- Control caving of roof in coal and ore excavations
- Enhance CO$_2$ sequestration
- Isolate toxic substances in rock

Thousands of treatments are successfully pumped each year

In natural conditions pressurized melted substance fractures earth crust leading to formation of veins of mineral deposits
**First Theoretical Models**

**KGD model; horizontal cross section**
- Khristianovich & Zheltov 1955
- Geertsma & de Klerk 1969

**PKN model; vertical cross section**
- Perkins & Kern 1961
- Nordgren 1972

Further Theoretical Work

**Studying of asymptotics and self-similar solutions**

Numerous papers on theoretical studying of hydraulic are focused on

(i) asymptotics at crack tip;
(ii) self-similar and asymptotic solutions to study regimes of flow

*Spence & Sharp 1985*: self-similar plane problem and asymptotics for newtonian liquid;

*Desrouches, Detournay et al 1994*: asymptotics for power-law liquid;

*Adachi & Detournay 2002*: self-similar plane problem for power-law liquid;

*Savitski & Detournay 2002*: self-similar axisymmetric problem for Newtonian liquid;

*Michell, Kuske & Pierce 2007*: asymptotics and regimes

*Hu & Garagash 2010*: plane problem; accounting for leak-off
Mathematical Formulation

Equations for Liquid

Fracture inlet \( x = 0 \)

Liquid front \( x^*(t) \)

Crack tip \( x_C(t) \)

Lag \( x_C - x^* \)

Continuity equation (mass conservation)

\[ \text{div} q + \frac{\partial w}{\partial t} - q_e = 0 \]  \hspace{1cm} (1)

Poiseuille equation (viscous flow in narrow channel)

\[ q = -D(w, p) \text{grad} p \]  \hspace{1cm} (2)

Reynolds equation (using (2) in (1))

\[ \text{div}[D(w, p) \text{grad} p] - \frac{\partial w}{\partial t} + q_e = 0 \]

Initial condition (zero opening)

\( w(x, 0) = 0 \)

Boundary condition (at liquid contour)

\( q_n(x) = q_0(x) \quad x \in L_q \quad p(x) = p_0(x) \quad x \in L_p \)

Global mass balance

\[ \frac{dV_e}{dt} = \int_S \left( \frac{\partial w}{\partial t} + \text{div} q \right) dS \]

The opening \( w \) being unknown, we need an equation for embedding solid (rock)
Mathematical Formulation

Equations for Solid

Fracture inlet \( x = 0 \)

Liquid front \( x^*(t) \)

Crack tip \( x_C(t) \)

Lag \( x_C - x^* \)

Solid mechanics equation
(commonly BIE of linear elasticity)

\[
A(w, p) = 0
\]

Boundary condition (at crack contour)

\[
w(x_C) = 0
\]

Fracture mechanics strength equation
(commonally in terms of SIFs)

\[
K_I = K_{Ic}
\]

Strength limitation permits crack propagation;
in general, it also defines the lag
between the liquid front and the crack tip
Simulators of Hydraulic Fractures

Planar fracture geometry based on rectangular boundary elements

Source elements

Interpolated front

\[ \Delta x \]

\[ \Delta y \]

Simulators

USA: Schlumberger (Siebrits et al)
USA: (Cleary et al)
Japan: (Jamamoto et al.)


The authors emphasize the need “to dramatically speed up … simulators”
Means to Meet Challenge

We need clear understanding of computational difficulties, which strongly influence the accuracy and stability of numerical results and robustness of procedures.

An appropriate means may be:
Using the methods developed in well-established THEORY OF PROPAGATING INTERFACES

The basic concept of these methods is **SPEED FUNCTION**

*To the date, it has not been employed for hydraulic fracture simulation*

*Further discussion explains the reasons “WHY NOT?”*
REVISITING FUNDAMENTALS

Eqn for time derivative of an integral over moving volume

\[
\frac{d}{dt} \int \rho(x,t)dV = \lim_{\Delta t \to 0} \frac{1}{\Delta t} \left\{ \int \rho(x,t+\Delta t)dV - \int \rho(x,t)dV \right\} = \int \frac{\partial \rho}{\partial t} dV + \int \rho v_n dS
\]

... = \int \frac{\partial \rho}{\partial t} dV + \int \rho v_n dS \quad v_n \text{ is the particle velocity at } S(t)

Then the mass conservation law reads

\[
\frac{dM_e}{dt} = \int \frac{\partial \rho}{\partial t} dV + \int \rho v_n dS \quad \text{MC}
\]

where now \( \rho(x,t) \) is the mass density, \( M_e \) is the external mass income

For incompressible homogeneous liquid \( \rho(x,t) = \text{const} \), \( M_e = \rho Ve \),

Volume conservation law (with obvious physical meaning):

\[
\frac{dVe}{dt} = \int v_n dS \quad \text{VC}
\]

Specify the VC for flow in a narrow channel
Speed Function and Speed Equation

**Volume conservation for flow in narrow channel**

\[
\frac{dV_e}{dt} = \int v_n dS \quad \text{VC}
\]

\[
S(t) = S^+ + S^- + S_L
\]

At \( S^+ \): \( v_n = \frac{\partial u^+_z}{\partial t} \)

At \( S^- \): \( v_n = -\frac{\partial u^-_z}{\partial t} \)

At \( S_L \): \( dS_L = w_L dL \)

**Hence for flow in narrow channel**

\[
\int v_n dS = \int \frac{\partial w}{\partial t} dS + \int w v_n dS
\]

where \( w = u^+_z - u^-_z \) is the channel width (opening)

**VC equation becomes**

\[
\frac{dV_e}{dt} = \int \frac{\partial w}{\partial t} dS + \int q_n dL
\]

Herein \( q_n = w v_n \) is the total flux through the opening \( w \)

Physical meaning is obvious. For a ‘rigid-wall’ channel, \( \frac{\partial w}{\partial t} = 0 \)
Speed Function and Speed Equation

**VC for entire liquid in narrow channel**

Thus for any volume of incompressible homogeneous liquid:

\[
\frac{dV_e}{dt} = \int S_m(t) \frac{\partial w}{\partial t} dS + \int q_n dL
\]

**VC**

Apply it to the *entire* volume \(V_t\) occupied by liquid at time \(t\)

\[
q_{n*} = w_\ast v_{n*} \quad \text{flux through the liquid front}
\]

The front Speed Equation is

\[
v_{n*} = \frac{dx_{n*}}{dt} = \frac{q_{n*}}{w_\ast}
\]

\(F = \frac{q_{n*}}{w_\ast} \quad \text{is the so-called Speed Function (SF)}\)

**Comment.** For 1-D case, VC yields \(\int w(x,t)dx = V_e(t)\) Then for ‘rigid’ channel, integration gives \(x_\ast(t) = 0_f(V_e(t))\) This implies that \(x_\ast = \infty\) at finite time \(t_\ast\) if width \(w(x)\) decreases fast enough.

Solution does not exist for \(t > t_\ast\).

This simple example indicates possible difficulties for a narrow channel.
Particular Forms of SE and SF: flow in narrow channel

NOTE THAT these forms of SE and SF are specific for hydraulic fracture: they appear ONLY because the channel is narrow what gives rise to the concept of the total flux $q$ through the width of the channel Respectively, only the total flux $q$ enters Poiseuille equation for viscous flow in a narrow channel

Comment. The particle velocity $v = q/w$ behind the front does not enter HF equations. Still its clear physical meaning makes it of value for an appropriate choice of unknown functions
Specification of Speed Function

for hydraulic fracture

Poiseuille eqn for flow of viscous liquid in narrow channel

\[ q = -D(w, p) \nabla p \]

Vectors \( q = (q_1, q_2) \)
\( \nabla = (\partial / \partial x_1, \partial / \partial x_2) \)

are defined in the channel tangent plane

This yields the speed equation for hydraulic fracture front

\[ v_{n*} = -\frac{1}{w_*} D(w, p) \frac{\partial p}{\partial n*} \]

SPEED EQUATION

for hydraulic fracture

\[ F = \frac{q_{n*}}{w_*} = -\frac{1}{w_*} D(w, p) \frac{\partial p}{\partial n*} \]

SPEED FUNCTION

for hydraulic fracture
Specification of Speed Equation for zero-lag case

Authors of computer simulators and commonly authors of papers on hydraulic fracture assume that there is NO LAG between the fluid front and the crack tip: \( x_\star = x_C \)

Thus both the opening and the flux are zero at the crack contour coinciding with the liquid front: \( w_\star = 0, \quad q_{n\star} = 0 \)

This results in the uncertainty \( v_{n\star} = \frac{q_{n\star}}{w_\star} = \frac{0}{0} \) what complicates using of SF

Perhaps this explains why the speed equation has not been used for simulation of hydraulic fracture to the date

Still the speed equation is to be met in limit:

\[
v_{n\star}(x_\star) = \lim_{x \to x_\star} \frac{q_n(x)}{w(x)} = \lim_{x \to x_\star} \left[ -\frac{1}{w(x)} D(w, p) \frac{\partial p}{\partial n_\star} \right]
\]
Particular Feature of Problem

for hydraulic fracture

Continuity equation (local form)
\[ \text{div} \mathbf{q} + \frac{\partial w}{\partial t} - q_e = 0 \]  
(1)

Poiseuille equation
\[ q = -D(w, p) \text{grad} p \]  
(2)

Reynolds equation (using (2) in (1))
\[ \text{div}[D(w, p) \text{grad} p] - \frac{\partial w}{\partial t} + q_e = 0 \]

Initial condition (zero opening)
\[ w(x,0) = 0 \]

BC from physical considerations (at the liquid contour)
\[ q_n(x) = q_0(x) \quad x \in L_q \quad p(x) = p_0(x) \quad x \in L_p \quad \text{BC} \]

But! We have additional SPEED EQUATION (at the liquid contour)
\[ v_n* = \frac{q*}{w*} = -\frac{1}{w*} D(w, p) \frac{\partial p}{\partial n*} \quad x \in L_q + L_p \quad \text{BC=SE} \]

Thus for the **elliptic** (in spatial coordinates) operator
we have **two** rather than one boundary conditions
involving a function and normal derivative.
This indicates that there might be difficulties.
Specifically, a problem might be *ill-posed*
Hadamard Definition and Tychonoff Regularization

Jacques Hadamard (1902), *Sur les problèmes aux dérivées partielles et leur signification physique*, Princeton University Bulletin 49-52

By Hadamard, a problem is well-posed when

- A solution exists
- The solution is unique
- The solution depends continuously on the data, in some reasonable metric

Otherwise, a problem is ill-posed

A.N. Tychonoff (1943) clearly recognized significance of ill-posed problems for applications. He was the first to suggest a means to solve them numerically by using regularization:

It looks reasonable to illustrate the specific features of the hydraulic fracture simulation by a clear example:

“The art of doing mathematics consists in finding that special case which contains all the germs of generality.”

*D. Hilbert*

Consider the Nordgren problem as “*that special case*”
- to evidently see that the problem is ill-posed and
- to find a proper method of regularization to have accurate and stable numerical results
Nordgren Problem formulation

Continuity equation (no leak-off)
\[ \frac{\partial q}{\partial x} + \frac{\partial w}{\partial t} = 0 \]

Poiseuille equation (Newtonian liquid)
\[ q = -k_l w^3 \frac{\partial p}{\partial x} \]

Reynolds equation (Newtonian liquid)
\[ \frac{k_l}{\partial x} \left( w^3 \frac{\partial p}{\partial x} \right) - \frac{\partial w}{\partial t} = 0 \]

Elasticity equation to find \( w \)
\[ p = k_r w \]

Then after using dimensionless variables, the problem becomes
\[ \frac{\partial^2 w^4}{\partial x^2} - \frac{\partial w}{\partial t} = 0 \] Nordgren PDE

\[ - \frac{\partial w^4}{\partial x} \bigg|_{x=0} = q_0 \] BC at inlet \( x = 0 \)

\[ w(x_*, t) = 0 \] BC at liquid front \( x = x_* \)

The solution should be such that:
\[ w(x, t) > 0 \quad 0 \leq x < x_* \], \quad w(x, t) = 0 \quad x > x_* \]
The speed equation (no-lag case) is

\[ q = -4 \frac{\partial w^3}{\partial x} \]

Then

\[ q = -4 \frac{\partial w^3}{\partial x} \]

and the speed equation becomes

\[ v_* = -4 \frac{\partial w^3}{\partial x} \bigg|_{x = x_*} \quad \text{SE} \]

The SE implies that the variable \( w^3 \) is preferable: its spatial derivative is finite. In terms of \( w^3 \), the problem is reformulated as

\[
\frac{\partial^2 w^3}{\partial x^2} + \frac{1}{3w^3} \left( \frac{\partial w^3}{\partial x} \right)^2 - \frac{1}{4w^3} \frac{\partial w^3}{\partial t} = 0 \quad \text{PDE}
\]

\[
\left. \frac{dw^3}{dx} \right|_{x=0} = -0.75 \frac{q_0}{3w^3(0)} \quad \text{BC at inlet } x = 0
\]

\[ w^3(x_*) = 0 \quad \text{BC at liquid front } x = x_* \]

\[ + \text{ SPEED EQUATION at the liquid front} \]
The problem is self-similar. Introduce automodel variables
\[ x = \xi t^{4/5}, \quad w(x) = t^{1/5} \psi (xt^{4/5}) \quad x_\ast = \xi_\ast t^{4/5} \]
Denote \( y(\xi) = \psi^3(\xi) \) The problem is reduced to ODE
\[
\frac{d^2 y}{d\xi^2} + a(y, dy/d\xi, \xi) \frac{dy}{d\xi} - \frac{3}{20} = 0 \quad \text{ODE}
\]
where \( a(y, dy/d\xi, \xi) = (dy/d\xi + 0.6\xi)/(3y) \) is finite at liquid front \( \xi = \xi_\ast \)

\[\left. \frac{dy}{d\xi} \right|_{\xi=0} = -0.75 \frac{q_0}{3y(0)} \quad \text{BC at inlet } x = 0 \]

\[ y(\xi_\ast) = 0 \quad \text{BC at liquid front } x = x_\ast \]

+ SPEED EQUATION at the liquid front
\[
\left. \frac{dy}{d\xi} \right|_{\xi=\xi_\ast} = -0.6\xi_\ast \quad \text{SE}
\]

Evidently, there are two BC at the liquid front
By direct substitution, it is easy to check that:

If \( y_1(\xi_1) \) is a solution for \( q = q_{01} \) with \( \xi_* = \xi_{*1} \) then
\[
y_2(\xi_2) = y_1(\xi_2 \sqrt{k}) / k
\]
is a solution for \( q_{02} = k^{-5/6} q_{01} \)
with \( \xi_* = \xi_{*1} / \sqrt{k} / x \)
\( k \) is an arbitrary positive number

This implies that there are two constants not depending on \( q_0 \):

\[
C_* = (q_0)^{0.6} / \xi_* \\
C_0 = y(0) / \xi_*^2
\]

**Conclusions:**

- Since
  \[
  \xi_* = (q_0)^{0.6} / C_*
  \]
it is a matter of convenience to prescribe \( q_0 \) or \( \xi_* \)

- A particular value of \( \xi_* \) may be also taken as convenient
Thus we have the problem for ODE
\[ \frac{d^2 y}{d\xi^2} + a(y, dy/d\xi, \xi) \frac{dy}{d\xi} - \frac{3}{20} = 0 \]  
ODE (1)

with conditions:
\[ \left. \frac{dy}{d\xi} \right|_{\xi=0} = -0.75 \frac{q_0}{3\sqrt{y(0)}} \]  
BC at inlet $\xi = 0$ (2)

\[ y(\xi_*) = 0 \]  
BC at liquid front $\xi_*$ (3)

\[ \left. \frac{dy}{d\xi} \right|_{\xi=\xi_*} = -0.6\xi_* \text{SE} = \text{BC at liquid front } \xi_* \] (4)

where $\xi_*$ is prescribed. For certainty, $\xi_* = 1$.

Hence, for ODE of second order, at the point $\xi_* = 1$, we have prescribed both the function and its derivative. Its solution defines $q_0$. Therefore, a small error in prescribing $q_0$ excludes the solution of the BV problem (1) - (3). By Hadamard definition, the BV problem (1) - (3) is ill-posed. IV problem (1), (3), (4) is well-posed.
Bench-Mark Solution
of well-posed initial value problem

Initial value (Cauchy) problem for ODE

\[
\frac{d^2 y}{d\xi^2} + a(y, dy/d\xi, \xi) \frac{dy}{d\xi} - \frac{3}{20} = 0
\]

ODE (1)

with initial conditions:

\[
y(0) = 0 \quad \text{BC at front} \quad (3)
\]
\[
\left. \frac{dy}{d\xi} \right|_{\xi = 0} = -0.6 \xi \quad \text{BC at front} \quad (4)
\]

The problem is solved by using R-K scheme of forth order

The bench-mark solution is obtained with 7 correct digits

\[
C_0 = 0.5820636
\]
\[
C_* = 0.7570913
\]

Values of \( \psi_1 = \sqrt{y_1} \) and \( d\psi_1 / d\xi \) are tabulated for \( \xi_* = \xi_{*1} = 1 \)

For the value \( q_0 = 2/\pi \) used by Nordgren, the benchmarks are

\[
\xi_* = 1.0073481 \quad \psi(0) = 0.8390284
\]

against the values given by Nordgren to the accuracy of about 1%

\[
\xi_* = 1.01 \quad \psi(0) = 0.83
\]

The bench-mark solution serves us to evaluate the accuracy of calculations without and with regularization
Solution of Self-Similar BV Problem

without regularization

Ill-posed BV problem:

\[
\frac{d^2 y}{d\xi^2} + a(y, dy/d\xi, \xi) \frac{dy}{d\xi} - \frac{3}{20} = 0 \quad \text{ODE (1)}
\]

with boundary conditions:

1. \( \frac{dy}{d\xi} |_{\xi=0} = -0.75 \frac{q_0}{\sqrt[3]{y(0)}} \) BC at inlet (2)
2. \( y(\xi) = 0 \) BC at front (3)

Up to 100,000 nodal points and up to 1500 iterations were used in attempts to reach the accuracy of three correct digits, at least

THE ATTEMPTS HAVE FAILED

By no means could we obtain more than two correct digits

The results always deteriorate near the liquid front

Comment. The accuracy of 1% is obtained even when using a rough mesh. Thus a rough mesh may serve for regularization when high accuracy is of no need. Still we need an appropriate regularization
Solution of Self-Similar BV Problem 

*with ε-regularization*

From the BC and SE at the front it follows that near the front: \[ y \approx 0.6\xi_* (\xi_* - \xi) \]

Hence instead of prescribing a BC at the front, we may impose it at a point at a small relative distance \( \varepsilon \) behind the front as

\[ y(\xi_\varepsilon) = 0.6\xi_*^2 \varepsilon \]

Now the problem is *well-posed*. It is solved by finite differences with iterations in non-linear terms and \( \xi_* \).

For \( \varepsilon = 10^{-3}, 10^{-4} \), the results for the step \( \Delta\xi = \Delta\xi / \xi_* = 10^{-3} - 10^{-6} \)

*coincide with the bench-mark solution*.

The essence of the suggested regularization consists in using the speed equation together with a prescribed BC to formulate the BC at a small relative distance \( \varepsilon \) behind the front rather than on the front itself.

*We call such an approach \( \varepsilon \) - regularization.*

Comment. For a coarse mesh the accuracy actually does not depend on the regularization parameter \( \varepsilon \).
Solution of Starting Problem \textit{without} regularization

When using as unknowns $w^4$ and $w^3$, the conclusions are same.

We solved the starting N-problem by using Crank-Nicolson scheme \textit{without regularization}.

By no means could we have more than two correct digits.

Similar to self-similar solution, fine meshes gave no improvement of the accuracy as compared with a rough mesh.

Similar to self-similar solution, fine meshes gave no improvement of the accuracy as compared with a rough mesh.

Nordgren PDE:
\[
\frac{\partial^2 w^4}{\partial x^2} - \frac{\partial w^4}{\partial t} = 0
\]
BC at inlet:
\[
\frac{\partial w^4}{\partial x} \bigg|_{x=0} = q_0
\]
BC at front:
\[
w(x^*, t) = 0
\]
Solution of Starting Problem

with $\varepsilon$ – regularization: change of spatial coordinate

$$\frac{\partial^2 w^3}{\partial x^2} + \frac{1}{3w^3} \left( \frac{\partial w^3}{\partial x} \right)^2 - \frac{1}{4w^3} \frac{\partial w^3}{\partial t} = 0$$

PDE

Introduce the coordinate moving with the front

$$\zeta = x / x_*(t)$$

Recalculate partial derivative $\frac{\partial \varphi}{\partial t} \bigg|_{x=\text{const}}$ to partial derivative $\frac{\partial \Phi}{\partial t} \bigg|_{\zeta=\text{const}}$ with $\Phi(\zeta, t) = \varphi(\zeta x_*(t), t)$

$$\frac{\partial \varphi}{\partial t} \bigg|_{x=\text{const}} = \frac{\partial \Phi}{\partial t} \bigg|_{\zeta=\text{const}} - \zeta \frac{x_*(t)}{\partial \zeta}$$

The PDF becomes

$$\frac{\partial^2 Y}{\partial \zeta^2} + A(Y, \frac{\partial Y}{\partial \zeta}, x_* v_* \zeta) \frac{\partial Y}{\partial \zeta} - B(Y, x_*) \frac{\partial Y}{\partial t} = 0$$

where

$$Y(\zeta, t) = w^3 (\zeta x_*(t), t) \quad A(Y, \frac{\partial Y}{\partial \zeta}, x_* v_* \zeta) = \frac{\partial Y}{\partial \zeta} + 0.75 x_* v_* \zeta \quad B(Y, x_*) = \frac{x_*^2}{4Y}$$

BC at inlet

$$\frac{4}{3} \frac{\partial Y}{\partial \zeta} \bigg|_{\zeta=0} = q_0$$

BC at front

$$Y(\zeta, t) \bigg|_{\zeta=1} = 0$$

SE at front

$$\frac{\partial Y}{\partial \zeta} \bigg|_{\zeta=1} = -0.75 x_* v_*$$

From conditions on the front it follows that near the front:

$$Y(\zeta, t) \approx 0.75 x_*(t) v_*(t)(1 - \zeta)$$
Solution of Starting Problem

**ε – regularization**

We have obtained that near the front:

\[ Y(\zeta, t) \approx 0.75x_*(t)v_*(t)(1 - \zeta) \]

Hence, now we may impose the BC at the relative distance \( \varepsilon \) behind the front

\[ Y(\zeta_\varepsilon, t) = 0.75x_*(t)v_*(t)\varepsilon \quad \text{BC near front} \]

We may expect that the problem is **well-posed** and provides the needed regularization. Extensive numerical tests confirm the expectation.

The problem is solved by using Crank-Nicolson scheme and iterations for non-linear multipliers and \( v_*(t) \).

For \( 10^{-2} > \varepsilon > 10^{-4} \) \( \Delta \zeta \leq 0.01 \) the results are accurate and stable in a wide range of the values of the time step and for very large number (up to 100 000) of steps. Error is less than 0.03%. There are no signs of instability in specially designed experiments.

The time expense on a conventional laptop does not exceed 15 s.

This implies that \( \varepsilon \) - regularization is efficient.
Conclusions

The *speed function* for fluid flow in a thin channel is given by the ratio of the total flux to width at the front. Its using facilitates employing level set methods and fast marching methods.

- The *speed equation* is a general condition at the liquid front *additional* to commonly formulated BC for hydraulic fracture.

- Using the SE extends options for numerical simulation of HF. It also indicates that the problem may be *ill-posed*.

- Suggested *ε - regularization* consists in employing the SE with a prescribed BC on the front to get a new BC at a small distance behind the front. It appears to be efficient.

- Studying the Nordgren problem evidently discloses the features of hydraulic fracture simulation. It gives the key to overcome the difficulty. Its solution provides the *bench-marks* useful for evaluating the accuracy.
Thank you!