ISSN 1028-3358, Doklady Physics, 2011, Vol. 56, No. 8, pp. 436–438. © Pleiades Publishing, Ltd., 2011.
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Speed Equation and Its Application for Solving Ill-Posed Problems of Hydraulic Fracturing[1]

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Presented by Academician N. F. Morozov March 17, 2011

Received March 31, 2011

DOI: 10.1134/S1028335811080015

Mathematical modeling of a fluid driven fracture, first discussed in [1], is of prime significance for hydraulic fracturing. Models developed to date employ the integral form of global mass balance (e.g. [2–5]). We demonstrate that using the local form, called the speed equation, shows specific features of the problem: it is ill-posed when considered as a boundary value (BV) problem. The equation also provides a means to regularize the problem and solve it efficiently.

Initially, we show that the speed equation is fundamental in the sense that it does not depend on a particular shear law of a liquid. When applied to a narrow channel between closely located boundaries, the mass conservation equation for an incompressible liquid is

$$\frac{dVe}{dt} = \int\limits_{S_l}\frac{\partial w}{\partial t}dS + \int\limits_{L_l(t)} w_*(x_*)\upsilon_{n*}(x_*)dL, \qquad (1)$$

where S_l is the middle surface, w is the height (opening) of the channel, $L_l(t)$ is the contour of the liquid front at the time t, x_* is a point on the front, υ_{n*} is the normal to L_l component of the fluid particle velocity averaged across the height. Note that in (1), the average particle velocity υ_{n*} also represents the speed of the front propagation. As $q_{n*}(x_*) = w_*(x_*)\upsilon_{n*}(x_*)$ is the flux through the front cross-section, we obtain the fundamental equation which gives the front velocity as a function of the flux and opening:

$$\upsilon_{n_*}(x_*) = \frac{q_{n_*}(x_*)}{w_*(x_*)}. \qquad (2)$$

[1] The article was translated by the authors.

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Use the Reynolds equation for flow of viscous incompressible liquid in a narrow channel:

$$\frac{\partial w}{\partial t} - \frac{\partial}{\partial x_i}\left(D(w,p)\frac{\partial p}{\partial x_i}\right) - q_e = 0, \qquad (3)$$

where D is a prescribed function or operator; p is the pressure, averaged through the cross-section; υ_i ($i = 1, 2$) are components of the average velocity of liquid particles in a channel cross section; the Cartesian coordinates x_1, x_2 are located in the fracture plane. Non opening fracture along a crack trajectory is assumed as an initial condition when studying hydraulic fracture. The boundary condition on the liquid front is the condition of the prescribed flux q_0 at a part L_q and of the prescribed pressure p_0 at the remaining part L_p of the contour L_l:

$$q_n(x) = q_0(x), \quad x \in L_q; \quad p(x) = p_0(x), \quad x \in L_p. \quad (4)$$

The opening in (3) being unknown, we need elasticity equation connecting the opening w and pressure p. Additionally, the criterion of linear fracture mechanics is imposed: $K_I = K_{IC}$, where K_I is the stress intensity factor, K_{IC} is its critical value.

In view of (2), prescribing the boundary conditions (4) means that there are two conditions at the points of a liquid front. This leads to difficulties common to over-determined problems [7–9] when solving the problem numerically, because the boundary is fixed on iteration. To find a means to overcome the difficulties, we study the Nordgren problem [2]. The Nordgren model considers straight fracture along the x-axis (figure) with the assumption that the pressure p is proportional to the opening w. Neglecting liquid leak-off and normalizing the variables, the Eq. (3) reads [2]:

$$\frac{\partial^2 w^4}{\partial x^2} - \frac{\partial w}{\partial t} = 0. \qquad (5)$$

The boundary conditions include the prescribed normalized flux q_0 at the inlet $x = 0$:

$$\frac{\partial w^4}{\partial x} = -q_0 \tag{6}$$

and zero opening (and flux) at the liquid front $x = x_*$, which coincides with the crack tip:

$$w(x_*) = 0. \tag{7}$$

The opening is assumed positive for $0 \le x < x_*$. We shall also use the speed Eq. (2) which becomes:

$$\upsilon_* = -\frac{4}{3}\frac{\partial w^3}{\partial x}\bigg|_{x=x_*}. \tag{8}$$

The problem being self-similar, the solution is represented as $w = t^{1/5}\psi(\varepsilon)$, where $\xi = xt^{-4/5}$, so that $x = \xi t^{4/5}$, $x_* = \xi_* t^{4/5}$, $\upsilon_* = \frac{dx_*}{dt} = 0.8\xi_* t^{-1/5}$, ξ_* is the automodel coordinate of the liquid front depending only on the prescribed flux q_0. Then the Eq. (5) becomes the ordinary differential equation:

$$\frac{d^2y}{d\xi^2} + a\left(y, \frac{dy}{d\xi}, \xi\right)\frac{dy}{\partial\xi} - \frac{3}{20} = 0, \tag{9}$$

where $y(\xi) = \psi^3(\xi)$, $a\left(y, \frac{dy}{d\xi}, \xi\right) = \frac{1}{3y}\left(\frac{dy}{d\xi} + 0.6\xi\right)$. The boundary conditions (6) and (7) read:

$$\frac{dy}{\partial\xi}\bigg|_{\xi=0} = -0.75\frac{q_0}{\sqrt[3]{y(0)}}, \tag{10}$$

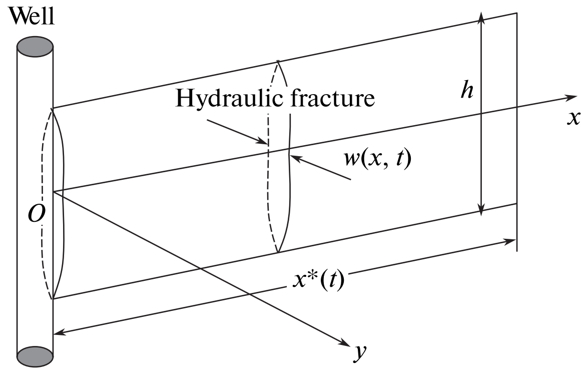
$$y(\xi_*) = 0, \tag{11}$$

and the speed Eq. (8) becomes:

$$\frac{dy}{\partial\xi}\bigg|_{\xi=\xi_*} = -0.6\xi_*. \tag{12}$$

It is easily shown that $C_* = \frac{q_0^{0.6}}{\xi_*}$ and $C_0 = \frac{y(0)}{\xi_*^2}$ are constants independent of the flux q_0. Since $\xi_* = \frac{q_0^{0.6}}{C_*}$, we may prescribe q_0 or ξ_*, as convenient. A particular value of q_0 or ξ_* may also be conveniently taken.

We can now fix ξ_*. Then according to (11), (12), at a fixed point ξ_* we have prescribed both the function y and its derivative $\frac{dy}{d\xi}$. Thus, for the equation of the second order (9) we have a Cauchy problem. Its solution defines $y(0)$ and $\frac{dy}{d\xi}\Big|_{\xi=0}$ and consequently the flux q_0 at $\xi = 0$. A small error when prescribing q_0 in (10) excludes the existence of the solution of the BV problem (9)–(11). By definition [7], the BV problem (9)–(11) is ill-posed and needs regularization [8, 9].



Scheme of the problem on hydraulic fracture propagation.

Conversely, the Cauchy problem (9), (11), (12) is well-posed and leads to a bench-mark solution. We obtained the solution by applying the fourth order Runge-Kutta scheme to the system of two differential equations in unknowns $y_1(\xi) = y(\xi)$, $y_2(\xi) = \frac{dy}{d\xi}$, equivalent to (9). The constants C_* and C_0 evaluated with seven significant digits are: $C_* = 0.7570913$, $C_0 = 0.5820636$. For the value $q_0 = 2/\pi$, used by Nordgren [2], we have $\xi_* = 1.0073486$, $\psi(0) = 0.8390285$ against the values $\xi_* = 1.01$, $\psi(0) = 0.83$ given by this author with the accuracy of about one percent. Bench-mark values of the function $y(\xi)$ and its derivative served us to evaluate the accuracy of further calculations obtained by using various approaches.

We could see that when solving the BV problem (9)–(11) it is impossible to obtain more than two correct digits. What is notable, this level of accuracy was obtained even when using a rough mesh with only one-hundred nodes. This implies that using a rough mesh may serve to regularize the problem when high accuracy is not needed. For fine meshes, we could see strong deterioration of the results near the liquid front $\xi = \xi_*$.

Likewise, our attempts to accurately solve the problem (5)–(7) also failed when using time steps with finite difference approximations for $\frac{\partial^2 w}{\partial x^2}$ and $\frac{\partial w}{\partial x}$ at a step. By no means could we have three correct digits, and the results always strongly deteriorated near the liquid front ??. Again, fine meshes did not improve the accuracy as compared with a rough mesh having the step $\Delta\varsigma = \frac{\Delta x}{x_*} = 0.01$.

The experiments confirm that the ill-posed problem under consideration cannot be solved accurately without regularization. A regularization method is suggested by the conditions (11), (12). Indeed, they yield the approximate Eq. $y \approx 0.6\xi_*(\xi_* - \xi)$ near the front. Hence, instead of prescribing a boundary condition at the front $\xi = \xi_*$, we impose it at a point $\xi_\varepsilon = \xi_*(1-\varepsilon)$ at a small relative distance ε from the front:

$$y(\xi_\varepsilon) = 0.6\xi_*^2\varepsilon. \tag{13}$$

The BV problem (9), (10), (13) is well-posed; it may be solved by finite differences. It appears that with $\varepsilon = 10^{-3}$, 10^{-4}, the results for the steps $\Delta\varsigma = \frac{\Delta\xi}{\xi_*} = 10^{-3}$, 10^{-4}, 10^{-5}, 10^{-6} coincided with those provided by the bench-mark solution. The results are stable if ε and $\Delta\varsigma$ are not simultaneously too small (ε, $\Delta\varsigma > 10^{-5}$). However, as expected, the results deteriorate when both ε and $\Delta\varsigma$ are too small; they become absolutely wrong when $\varepsilon = \Delta\varsigma = 10^{-6}$. We could also see that as ε increases, the accuracy decreases and it actually does not depend on the step if the latter is small enough. In particular, for the step $\Delta\varsigma = 0.1$, the accuracy is one percent for $\varepsilon = 0.01$, and the results stay at the same accuracy level even for $\varepsilon = 10^{-9}$.

The suggested regularization consists in using the speed equation together with a prescribed boundary condition to formulate the boundary condition at a small relative distance ε behind the front rather than on the front itself. We call such an approach ε-regularization. It is applicable in general 1D and 2D cases when a self-similar formulation is not available or is not used. To illustrate, we employed the ε—regularization for the starting Eq. (5) under the boundary conditions (6), (7). In terms of the variable $Y = w^3$, the prescribed condition (7) and the speed Eq. (8) yield

$$Y(x,t) \approx 0.75x_*(t)\upsilon_*(t)\left[1 - \frac{x}{x_*(t)}\right]$$

at points close to the front. Hence, the boundary condition at a point $x_\varepsilon = x_*(1 - \varepsilon)$ with the relative distance ε from the front is:

$$Y(x_\varepsilon, t) = 0.75x_*(t)\upsilon_*(t)\varepsilon. \tag{14}$$

Thus, the regularized problem consists in solving (5) under zero-opening initial condition and the boundary conditions (6) and (14). Numerical experiments have shown that the ε-regularization removes the difficulties and provides accurate results.

The conclusions of the paper are as follows: (i) the derived speed equation may serve for tracing hydraulic fracture by methods of the theory of propagating surfaces; (ii) when simulating hydraulic fracture numerically, it is useful to employ the ε—regularization consisting in prescribing a boundary condition at a small relative distance s behind the front; (iii) the method provides an efficient means for solving problems of hydraulic fracture.

ACKNOWLEDGMENTS

The author appreciates the support of the EU Marie Curie IAPP program (Grant no. 251475).

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